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Cut Points and Diffusions in Random Environment

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Abstract In this article we investigate the asymptotic behavior of a new class of multidimensional diffusions in random environment. We introduce cut times in the spirit of the work done by Bolthausen et al. (Ann. Inst. Henri Poincaré 39(5):527–555, 2003) in the discrete setting providing a decoupling effect in the process. This allows us to take advantage of an ergodic structure to derive a strong law of large numbers with possibly vanishing limiting velocity and a central limit theorem under the quenched measure.

Keywords Cut points · Diffusions in random environment · Quenched invariance principle · Law of large numbers · Diffusive behavior

Mathematics Subject Classification (2000) 82D30 · 60K37

1 Introduction

The object of this article is to introduce a special class of multidimensional diffusions in random environment for which we are able to prove a law of large numbers and a functional central limit theorem governing the corrections to the law of large numbers, valid for a.e. environment (a so-called *quenched* functional central limit theorem). The investigation of the asymptotic behavior of multidimensional diffusions in random environment is well known for its difficulty due to the massively non-self-adjoint character of the model and to the rarity of explicitly calculable examples. A special interest of the class we introduce stems from the fact that it offers examples of diffusions with nonvanishing random drifts where, on the one hand, no invariant

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measure for the process of the environment viewed from the particle, absolutely continuous with respect to the static distribution of the random environment, is known and where, on the other hand, our results hold without certain assumptions which guarantee condition (T) or (T′) of Sznitman, see [31, 32, 34] in the discrete set-up and [10, 26, 27] in the continuous set-up. Thus, when the limiting velocity vanishes, such examples correspond to diffusive motions where very few results are available, see [5] and [4] in the discrete set-up, or [36] for diffusions in random environment; when the limiting velocity does not vanish, such examples differ from existing results for *quenched* functional central limit theorems, such as in the recent [1] or [24] in the discrete set-up, since such results rely on finiteness assumptions for moments of certain regeneration times, which, to the best of our knowledge, can only be checked through some sufficient criterion for (T) or (T′). Let us mention that at present it is an open problem whether ballistic behavior in dimension 2 and above implies (T) or (T′). Our class contains examples of ballistic motion, and we do not need to check (T) or (T′) (which of course does not preclude that these conditions may hold in these examples). The class we introduce here is a type of continuous counterpart of the class considered in [4] in the context of random walks in random environment. The formulas we obtain for the velocity are reasonably explicit and might be amenable to the construction of some further examples or counterexamples, in the spirit of what was done in [4], although this is not carried out here given the length of this work. Indeed the continuous set-up is more delicate than the discrete set-up, and it is by no mean routine to adapt the general strategy of [4] in the context of diffusions in random environment. For an overview of results and useful techniques concerning this area of research, we refer to [33, 35, 39] and, in particular, to [14–16, 23, 24] for recent advances with the help of the *method of the environment viewed from the particle.*

Before describing our results further, let us first introduce the model. We consider integers $d_1 \geq 5$, $d_2 \geq 1$, and $d = d_1 + d_2$. The random environment is described by a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, and we assume the existence of a group $\{\tau_x : x \in \mathbb{R}^d\}$ of $\mathbb{P}$-preserving transformations on Ω that are jointly measurable in x and ω. On $(\Omega, \mathcal{A}, \mathbb{P})$, we consider an $\mathbb{R}^d$-valued random variable $b(\cdot)$ with vanishing first d_1 components, that is,

$$b(\omega) = \big(\underbrace{0, \ldots, 0}_{d_1}, b^*(\omega)\big) \in \mathbb{R}^d \quad \text{for } \omega \in \Omega, \tag{1.1}$$

and we define

$$b(x, \omega) \overset{\text{def}}{=} b\big(\tau_x(\omega)\big) \quad \text{for } x \in \mathbb{R}^d. \tag{1.2}$$

We assume this function to be bounded and Lipschitz continuous, i.e., there is a constant $\kappa > 0$ such that, for all $x, y \in \mathbb{R}^d$ and $\omega \in \Omega$,

$$\big|b(x, \omega)\big| \leq \kappa, \qquad \big|b(x, \omega) - b(y, \omega)\big| \leq \kappa |x - y|, \tag{1.3}$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^d$. We will further assume finite range dependence for the environment, that is, for a Borel subset F of $\mathbb{R}^d$, we define the σ-algebra

$$\mathcal{H}_F \overset{\text{def}}{=} \sigma\big(b(x, \omega) : x \in F\big) \tag{1.4}$$

and assume that there is $R > 0$ such that

$$\mathcal{H}_A \text{ and } \mathcal{H}_B \text{ are independent whenever } d(A, B) > R, \tag{1.5}$$

where $d(A, B) = \inf\{|x - y| : x \in A,\ y \in B\}$. We let stand $(X_t)_{t\ge 0}$ for the canonical process on $C(\mathbb{R}_+, \mathbb{R}^d)$ and, for $\omega \in \Omega$, $x \in \mathbb{R}^d$, we denote by $P_{x,\omega}$ the unique solution to the martingale problem attached to x and

$$\mathcal{L}^\omega = \frac{1}{2}\Delta + b(\cdot, \omega)\cdot\nabla, \tag{1.6}$$

i.e., the law $P_{x,\omega}$ describes the diffusion in the environment ω starting at x and is usually called the *quenched law*. We write $E_{x,\omega}$ for the corresponding expectation. We endow the space $C(\mathbb{R}_+, \mathbb{R}^d)$ with the Borel σ-algebra $\mathcal{F}$ and the canonical right-continuous filtration $(\mathcal{F}_t)_{t\ge 0}$. For the study of the asymptotic properties of $X_\cdot$, it is convenient to introduce the *annealed law* which is the semi-direct product measure on $\Omega \times C(\mathbb{R}_+, \mathbb{R}^d)$ defined as

$$P_x \stackrel{\text{def}}{=} \mathbb{P} \times P_{x,\omega}. \tag{1.7}$$

We denote with E_x the corresponding expectation. Let us mention that the laws P_x typically destroy the Markovian structure but restore a useful stationarity to the problem.

Let us now explain the purpose of this work in more detail. In the first part of this article, we prove a law of large numbers, see Theorem 2.11, namely, when $d_1 \ge 5$, we show that

$$P_0\text{-a.s.,} \qquad \frac{X_t}{t} \longrightarrow v \stackrel{\text{def}}{=} E^{P\times K_0}\left[\int_0^{T^1} b(\chi_u, \omega)\,du,\ T^0 = 0\right] \quad \text{as } t \to \infty, \tag{1.8}$$

with a deterministic (possibly vanishing) limiting velocity v. The process χ_u, $u \ge 0$, is defined on an enlarged probability space, see Theorem 2.2, on which the notion of doubly infinite bilateral cut times $(T^k,\ k \in \mathbb{Z})$ for the Brownian part of the diffusion is superimposed, see (2.24). The definition of cut times involves neither the drift nor the random environment except for the parameters κ of (1.3) and R of (1.5). The law of $(\omega, (\chi_u)_{u\ge 0})$ under the measure $P \times K_0$ recovers the *annealed* measure P_0, see (2.12), (1) of Theorem 2.2 and (2.17).

In the second part, assuming the antipodal symmetry in the last d_2 components of the drift, see (3.1), and when $d_1 \ge 7$ (in which case $v = 0$) or when $d_1 \ge 13$ without symmetry properties, we derive a functional central limit theorem under the *quenched law,* see Theorem 4.1:

> for $\mathbb{P}$-a.e. ω, under the measure $P_{0,\omega}$, the $C(\mathbb{R}_+, \mathbb{R}^d)$-valued random variables $B^r_\cdot \stackrel{\text{def}}{=} r^{-1/2}(X_{r\cdot} - vr\cdot)$, $r > 0$, where v corresponds to the limiting velocity in (1.8), converge weakly to a Brownian motion with deterministic covariance matrix as r tends to infinity. (1.9)

The proofs of the above results are based on the existence of so-called cut times T^k, $k \in \mathbb{Z}$, which are defined in a similar spirit to [4] and play a role comparable to the regeneration times introduced in [37]. The assumption $d_1 \geq 5$ enables one to exploit the presence of these cut times and discover a decoupling effect, see Proposition 2.8. The cut times are in essence defined as follows. In the spirit of the technique applied in [6] for random walks in random environments or in [28] for the continuous case, we couple our diffusion at each integer time n with an auxiliary Bernoulli variable Λ_n such that when $\Lambda_n = 1$, the distribution of X_{n+1} given X_n does not depend on the environment. We then say that a cut time occurs at an integer time n if the Bernoulli variable at time $n-1$ takes value 1 and if the future of the Brownian part of the diffusion, which corresponds to the first d_1 components, after time n stays at a distance at least $2R$ from the past before time $n-1$, see (2.24) and (2.28) for the exact definition. Due to the finite range dependence, see (1.5), we then can produce decoupling in our process which allows an easy comparison to a process defined on a probability space with an ergodic shift in which we can embed an additive functional. These considerations essentially reduce the proof of (1.8) to an application of Birkhoff's Ergodic Theorem. With the help of a criterion introduced by Bolthausen and Sznitman in [3], see Lemma 4, the *quenched* invariance principle (1.9) follows from the *annealed* versions, see Theorems 3.3 and 3.8, by a variance calculation which involves a certain control on the intersections of two independent paths. The main strategy behind the proofs of the *annealed* central limit theorems is to show an *annealed* central limit theorem for a process defined as the polygonal interpolation of an ergodic process $Z^s_k, k \in \mathbb{Z}$, see (2.48) and Proposition 2.10, which is then rescaled in time and space analogously to the definition of $B^n_\cdot$ in (1.9) for integers $n \geq 1$ and which is comparable to the original diffusion $X_\cdot$, see Lemma 3.5 and (3.46). The proof without symmetry assumption on the drift but $d_1 \geq 13$ is more involved and needs an adaptation of Gordin's method, see, for instance, the proof of Theorem 7.6 in [8].

Let us mention that the application of Girsanov's formula yields a very handy and reasonably explicit version of the transition density for the last d_2 components of the diffusion in a fixed environment given the Brownian part (first d_1 components), see (2.6). Formula (2.6) involves the Brownian transition density and the bridge measure which depend neither on the environment nor on the first d_1 components of the diffusion and hence enables to inspect the *quenched* transition density directly. This formula is not available anymore if one wants to treat more general diffusions in random environment where the diffusion matrix in the last d_2 components becomes a genuinely environment dependent stationary process. Other methods would be required in this set-up, possibly in the spirit of filtering theory.

Let us now explain how this article is organized. In Sect. 2, we couple our diffusion with a suitable sequence of i.i.d. Bernoulli variables, see Theorem 2.2. We then define the cut times T^k, $k \in \mathbb{Z}$, see (2.24) and (2.28), and provide the crucial decoupling, see Proposition 2.8. Finally, we prove a law of large numbers. Section 3 is dedicated to two central limit theorems under the annealed measure that are also consequences of the decoupling technique discussed in Sect. 2. The first central limit theorem is proved under a symmetry assumption on the drift and $d_1 \geq 7$, see (3.1), whereas, for the second central limit theorem, $d_1 \geq 13$ is assumed. In Sect. 4, we show how one can strengthen the results of Sect. 3 into central limit theorems under the *quenched* measure. Finally, in the Appendix, two multidimensional versions of central limit theorems for martingales are proved.

Convention on constants Unless otherwise stated, constants only depend on the quantities d_1, d_2, κ, R. In calculations, generic constants are denoted by c and may change from line to line, whereas $c_1, c_2, \ldots$ are constants with fixed values at their first appearance. By $c(q,\eta)$ we denote constants that depend on the usual parameters d_1, d_2, κ, R and additionally on q and η.

2 Decoupling and a Law of Large Numbers

In this section, we will first take advantage of the special structure of the diffusions considered in our model to couple auxiliary i.i.d. Bernoulli variables Λ_n with the diffusion, see Theorem 2.2. Under the coupled measure, the distribution of the diffusion at integer time n will only depend on the position at time $n-1$ and not on the environment when $\Lambda_{n-1} = 1$. Due to the finite range dependence, see (1.5), we then discover with the help of cut times, which are introduced in Sect. 2.2, a certain decoupling effect under the annealed law, see Proposition 2.8. This finally leads to a law of large numbers, see Theorem 2.11.

For a real number $u \in \mathbb{R}$, we define its integer part as

$$[u] \stackrel{\text{def}}{=} \sup\{n \in \mathbb{Z} \mid n \le u\}. \tag{2.1}$$

Further, we denote the d_2-dimensional closed ball of radius $r > 0$ centered at $y \in \mathbb{R}^{d_2}$ with $B_r^{d_2}(y)$ and write $\mathrm{vol}(d_2)$ for its volume. For $n \ge 1$, $z, z' \in \mathbb{R}^n$, and $s > 0$, we introduce the n-dimensional Gaussian kernel

$$p_n(s, z, z') \stackrel{\text{def}}{=} \frac{1}{(2\pi s)^{n/2}} \exp\{-|z - z'|^2/2s\}. \tag{2.2}$$

We denote by $W_0^{d_1}$ the set of all continuous $\mathbb{R}^{d_1}$-valued functions on $\mathbb{R}$ that vanish at 0. Furthermore, we consider the space $W_+^{d_2} = C(\mathbb{R}_+, \mathbb{R}^{d_2})$ and the canonical coordinate processes $X^1_\cdot, X^2_\cdot$ defined as

$$\begin{aligned} X_t^1(w) &\stackrel{\text{def}}{=} w(t) \quad \text{for all } t \in \mathbb{R} \text{ and } w \in W_0^{d_1}, \\ X_t^2(u) &\stackrel{\text{def}}{=} u(t) \quad \text{for all } t \ge 0 \text{ and } u \in W_+^{d_2}. \end{aligned} \tag{2.3}$$

We endow the space $W_0^{d_1}$ with the σ-algebra $\mathcal{W}_0 = \sigma(X_s^1, s \in \mathbb{R})$ and $W_+^{d_2}$ with the σ-algebra $\mathcal{U} = \sigma(X_s^2, s \in \mathbb{R}_+)$ and the canonical filtration $\mathcal{U}_t = \sigma(X_s^2, 0 \le s \le t)$, $t \ge 0$, which is neither right-continuous nor complete in opposition to $\mathcal{F}_t$, see above (1.7). $\bar{P}$ denotes the two-sided Wiener measure on $(W_0^{d_1}, \mathcal{W}_0)$ with $\bar{P}[X_0^1 = 0] = 1$. We write $\bar{E}$ for the expectation with respect to the measure $\bar{P}$. On the measurable space $(W_+^{d_2}, \mathcal{U})$, we introduce for $y, y' \in \mathbb{R}^{d_2}$ the Wiener measure $\tilde{P}_y$ with $\tilde{P}_y[X_0^2 = y] = 1$ and $\tilde{P}_{y,y'}$, the Brownian bridge measure from y to y' on [0,1]. We write $\tilde{E}_y$ and $\tilde{E}_{y,y'}$ for the corresponding expectations. On the product space $(W_0^{d_1} \times W_+^{d_2}, \mathcal{W}_0 \otimes \mathcal{U})$, we define the $\mathbb{R}^d$-valued process

$$\chi_t \stackrel{\text{def}}{=} (X_t^1, X_t^2), \quad t \ge 0. \tag{2.4}$$

For $y \in \mathbb{R}^{d_2}$, $\omega \in \Omega$, and $w \in W_0^{d_1}$, we denote by $\bar{K}_{y,\omega}(w)$ the probability kernel from $(W_0^{d_1}, \mathcal{W}_0)$ to $(W_+^{d_2}, \mathcal{U})$ defined as the unique solution of the martingale problem starting at time 0 from y and attached to

$$\mathcal{L}_t^{w,\omega} = \frac{1}{2}\sum_{i=1}^{d_2} \partial_{ii}^2 + \sum_{i=1}^{d_2} b_i^*\big(\big(w(t), \cdot\big), \omega\big)\partial_i,$$

see Theorem 6.3.4 in [30]. For $w \in W_0^{d_1}$ and $\omega \in \Omega$, we define the stochastic exponential

$$\mathcal{E}(w,\omega) \stackrel{\text{def}}{=} \exp\bigg\{\int_0^1 b^*\big(\big(w(s), X_s^2\big), \omega\big)\, dX_s^2 - \frac{1}{2}\int_0^1 \big|b^*\big(\big(w(s), X_s^2\big), \omega\big)\big|^2\, ds\bigg\}, \tag{2.5}$$

which is in $L^1(\tilde{P}_{y,y'})$ for all $y, y' \in \mathbb{R}^{d_2}$, see the proof of Theorem 4.1 of [19], and introduce the transition density

$$p_{w,\omega}(1,y,y') \stackrel{\text{def}}{=} p_{d_2}(1,y,y')\tilde{E}_{y,y'}\big[\mathcal{E}(w,\omega)\big], \tag{2.6}$$

which is a measurable function of $\omega \in \Omega$, $w \in W_0^{d_1}$, and $y, y' \in \mathbb{R}^{d_2}$, see for instance Theorem 44 on p. 158 in [22], fulfilling

$$\bar{K}_{y,\omega}(w)\big[X_1^2 \in G\big] = \int_G dy'\, p_{w,\omega}(1,y,y')$$

for all Borel sets G in $\mathbb{R}^{d_2}$, see (6.35) in Chap. 5 of [13] and Girsanov's Formula in Theorem 6.4.2 of [30].

2.1 The Coupling Construction

We are going to enlarge the probability space $(W_0^{d_1} \times W_+^{d_2}, \mathcal{W}_0 \otimes \mathcal{U}, \bar{P} \times \bar{K}_{y,\omega})$ and provide a coupling of the process $\chi_.$ with a sequence of i.i.d. Bernoulli variables, see Theorem 2.2. Let us begin with an easy fact about the transition density defined in (2.6), which will be crucial in the construction of our coupling.

Lemma 2.1 *Under the assumptions* (1.1) *and* (1.3) *on the drift* $b(\cdot)$, *there is a constant* $\varepsilon \in (0,1)$ *such that, for all* $\omega \in \Omega$, $w \in W_0^{d_1}$, $y \in \mathbb{R}^{d_2}$, *and* $y' \in B_1^{d_2}(y)$, *the following holds*:

$$p_{w,\omega}(1,y,y') > \frac{2\varepsilon}{\mathrm{vol}(d_2)}. \tag{2.7}$$

Proof By Jensen's inequality and (1.3) we obtain that $p_{\omega,w}(1,y,y')$ is greater than or equal to

$$e^{-\kappa^2/2} p_{d_2}(1,y,y') \exp\bigg\{\tilde{E}_{y,y'}\bigg[\int_0^1 b^*\big(\big(w(s), X_s^2\big), \omega\big)\, dX_s^2\bigg]\bigg\}. \tag{2.8}$$

Note that, under the measure $\tilde{P}_{y,y'}$, the process $X_t^2, t \in [0,1]$, is a Brownian bridge from y to y' in time 1 and therefore satisfies the following stochastic differential equation, see Example 5.6.17 (i) and p. 354 in [13],

$$\begin{cases} dX_t^2 = d\beta_t + \frac{y'-X_t^2}{1-t}\,dt, & 0 \le t < 1, \\ X_0^2 = y, & \tilde{P}_{y,y'}\text{-a.s.}, \end{cases} \tag{2.9}$$

for a d_2-dimensional standard Brownian motion $\beta_\cdot$. Thus,

$$\begin{aligned} \left|\tilde{E}_{y,y'}\left[\int_0^1 b^*\big((w(s),X_s^2),\omega\big)\,dX_s^2\right]\right| &= \left|\tilde{E}_{y,y'}\left[\int_0^1 b^*\big((w(s),X_s^2),\omega\big)\frac{y'-X_s^2}{1-s}\,ds\right]\right| \\ &\le \tilde{E}_{y,y'}\left[\kappa\int_0^1 \left|\nabla_{X_s^2}\log p_{d_2}\big(1-s,X_s^2,y'\big)\right|ds\right] \\ &\le c\exp\big\{-|y-y'|^2/c\big\}\,p_{d_2}^{-1}(1,y,y') \\ &\stackrel{\text{def}}{=} g_{d_2}(y-y'), \end{aligned}$$

where the last inequality follows from a result of [19], see Theorem 2.4. It is obvious that

$$B_1^{d_2}(0) \ni z \longmapsto g_{d_2}(z)$$

is a bounded map, and thus (2.8) is bounded away from 0 for all $y' \in B_1^{d_2}(y)$. This finishes the proof. $\square$

Before providing the construction of the coupling, let us introduce some further notation. We denote by $\Lambda_\cdot = (\Lambda_j)_{j\in\mathbb{Z}}$ the canonical coordinate process on $\{0,1\}^{\mathbb{Z}}$ and by $\mathcal{S}$ the canonical product σ-algebra generated by $\Lambda_\cdot$. We write $\lambda_\cdot = (\lambda_j)_{j\in\mathbb{Z}}$ for an element of $\{0,1\}^{\mathbb{Z}}$ and by $\mathbf{\Lambda}^\varepsilon$, where ε comes from (2.7), we denote the unique probability measure on $(\{0,1\}^{\mathbb{Z}},\mathcal{S})$ under which $\Lambda_\cdot$ becomes a sequence of i.i.d. Bernoulli random variables with success parameter ε. We also introduce the shift operators $\{\theta_m : m\in\mathbb{Z}\}$ and $\{s_t : t\ge 0\}$ operating on $(W_0^{d_1}\times\{0,1\}^{\mathbb{Z}},\mathcal{W}_0\otimes\mathcal{S})$ and $(W_+^{d_2},\mathcal{U})$ respectively such that

$$\theta_m(w,\lambda_\cdot) = \big(w(m+\cdot)-w(m),\lambda_{m+\cdot}\big), \tag{2.10}$$

$$s_t(u) = u(t+\cdot). \tag{2.11}$$

Note that the pair $(w,\lambda_\cdot)\in W_0^{d_1}\times\{0,1\}^{\mathbb{Z}}$ stands for the pair of processes $((w(t))_{t\in\mathbb{R}}, (\lambda_j)_{j\in\mathbb{Z}})$ with different parameter sets. On the product space $(W_0^{d_1}\times\{0,1\}^{\mathbb{Z}},\mathcal{W}_0\otimes\mathcal{S})$, we define the product measure

$$P \stackrel{\text{def}}{=} \bar{P}\otimes\mathbf{\Lambda}^\varepsilon, \tag{2.12}$$

recalling that $\bar{P}$ denotes the two-sided Wiener measure on $W_0^{d_1}$ with $\bar{P}[X_0^1=0]=1$.

Theorem 2.2 *There exists a probability kernel from* $\mathbb{R}^{d_2} \times \Omega \times W_0^{d_1} \times \{0,1\}^{\mathbb{Z}}$ *to* $W_+^{d_2}$, *which we denote by* $K_{y,\omega}(w,\lambda_\cdot)[O]$ *for* $y \in \mathbb{R}^{d_2}$, $\omega \in \Omega$, $w \in W_0^{d_1}$, $\lambda_\cdot \in \{0,1\}^{\mathbb{Z}}$, *and* $O \in \mathcal{U}$, *such that*:

(1) *For* $(y,\omega,w,\lambda_\cdot) \in \mathbb{R}^{d_2} \times \Omega \times W_0^{d_1} \times \{0,1\}^{\mathbb{Z}}$, *under the measure* $P \times K_{y,\omega}(w,\lambda_\cdot)$, *the process* $(\chi_t)_{t\ge 0}$ *is* $P_{(0,y),\omega}$*-distributed, where* $(0,y) \in \mathbb{R}^d$ *and, in particular,* W_t *which is defined by*

$$W_t \overset{\text{def}}{=} \chi_t - (0,y) - \int_0^t b(\chi_s,\omega)\,ds, \quad t \ge 0, \tag{2.13}$$

is a d*-dimensional standard Brownian motion in its own filtration on* $W_0^{d_1} \times \{0,1\}^{\mathbb{Z}} \times W_+^{d_2}$ *endowed with the probability* $P \times K_{y,\omega}(w,\lambda_\cdot)$.

(2) *For each integer* $n \ge 0$, $(y,\omega,w,\lambda_\cdot) \in \mathbb{R}^{d_2} \times \Omega \times W_0^{d_1} \times \{0,1\}^{\mathbb{Z}}$, *and any bounded measurable function* f *on* $W_+^{d_2}$, $K_{y,\omega}(w,\lambda_\cdot)$*-a.s.*,

$$E^{K_{y,\omega}(w,\lambda_\cdot)}\big[f\big(X_\cdot^2\big)\circ s_n \mid \mathcal{U}_n\big] = E^{K_{X_n^2,\hat\omega}(\theta_n(w,\lambda_\cdot))}\big[f\big(X_\cdot^2\big)\big] \tag{2.14}$$

with $\hat\omega = \tau_{(w(n),0)}(\omega)$. *Moreover,*

$$E^{K_{y,\omega}(w,\lambda_\cdot)}\big[f\big(X_\cdot^2\big)\big] = E^{K_{0,\tilde\omega}(w,\lambda_\cdot)}\big[f\big(y + X_\cdot^2\big)\big] \tag{2.15}$$

with $\tilde\omega = \tau_{(0,y)}(\omega)$.

(3) *For each* $(y,\omega,w,\lambda_\cdot) \in \mathbb{R}^{d_2} \times \Omega \times W_0^{d_1} \times \{0,1\}^{\mathbb{Z}}$ *with* $\lambda_0 = 1$, *we have that, under the probability measure* $K_{y,\omega}(w,\lambda_\cdot)$, X_1^2 *is uniformly distributed on the ball* $B_1^{d_2}(y)$.

(4) *For each integer* $n \ge 0$, $(z_1,z_2) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, $(y,\omega,w,\lambda_\cdot) \in \mathbb{R}^{d_2} \times \Omega \times W_0^{d_1} \times \{0,1\}^{\mathbb{Z}}$, *and any bounded measurable function* f *on* $W_+^{d_2} \times C(\mathbb{R}_+,\mathbb{R}^d)$, *we have that, for* $\bar\omega = \tau_{(z_1,z_2)}(\omega)$,

$$E^{K_{y,\bar\omega}(w,\lambda_\cdot)}\big[f\big(\big(X_\cdot^2, b(\chi_\cdot,\bar\omega)\big)_{\cdot\wedge n}\big)\big] \quad \text{is } \mathcal{H}_{(z_1+w([0,n]))\times\mathbb{R}^{d_2}}\text{-measurable}. \tag{2.16}$$

In order to shorten the notation, we will usually not write explicitly the dependence of the kernels on $(w,\lambda_\cdot) \in W_0^{d_1} \times \{0,1\}^{\mathbb{Z}}$, i.e., for $y \in \mathbb{R}^{d_2}$, $\omega \in \Omega$, we write $K_{y,\omega}$ instead of $K_{y,\omega}(w,\lambda_\cdot)$. In this sense, for a fixed $y \in \mathbb{R}^{d_2}$, we define the *annealed kernel* from $(W_0^{d_1} \times \{0,1\}^{\mathbb{Z}}, \mathcal{W}_0 \otimes \mathcal{S})$ to $(W_+^{d_2} \times \Omega, \mathcal{U} \otimes \mathcal{A})$ by

$$K_y \overset{\text{def}}{=} \mathbb{P} \times K_{y,\omega}. \tag{2.17}$$

Proof Given a probability kernel $K_{y,\omega}^{(\lambda)}(w)[O]$ for $O \in \mathcal{U}_1$, $w \in W_0^{d_1}$, $\lambda \in \{0,1\}$, $y \in \mathbb{R}^{d_2}$, and $\omega \in \Omega$, which will be specified in (2.20) below, there is a unique probability measure $K_{y,\omega}(w,\lambda_\cdot)$ on $\mathcal{U}$ for $w \in W^{d_1}$, $\lambda_\cdot \in \{0,1\}^{\mathbb{Z}}$, $y \in \mathbb{R}^{d_2}$, and $\omega \in \Omega$ such that, for integer $m \ge 1$ and $O \in \mathcal{U}_1$, $K_{y,\omega}(w,\lambda_\cdot)$-a.s.,

$$_{y,\omega}(w,\lambda_\cdot)\big[s_m^{-1}(O) \mid \mathcal{U}_m\big] = K^{(\lambda_m)}_{X_m^2,\tau_{(w(m),0)}(\omega)}\big(w(m+\cdot) - w(m)\big)[O]. \tag{2.18}$$

An application of Girsanov's Theorem, see, for instance, Theorem 6.4.2 of [30] and (6.35) in Chap. 5 of [13], shows that, for $w \in W_0^{d_1}$, $y \in \mathbb{R}^{d_2}$, $\omega \in \Omega$, and $O \in \mathcal{U}_1$, see below (2.4) and (2.6),

$$\bar{K}_{y,\omega}(w)[O] = \tilde{E}_y\big[\mathcal{E}(w,\omega), O\big] = \int_{\mathbb{R}^{d_2}} dy' p_{w,\omega}(1,y,y') \frac{\tilde{E}_{y,y'}[\mathcal{E}(w,\omega), O]}{\tilde{E}_{y,y'}[\mathcal{E}(w,\omega)]},$$

and so we define, for $\lambda \in \{0,1\}$ and $y' \in \mathbb{R}^{d_2}$,

$$h(w,\lambda,y,y',\omega) \stackrel{\text{def}}{=} \begin{cases} \frac{\mathbb{1}\{y' \in B_1^{d_2}(y)\}}{\operatorname{vol}(d_2)} & \text{if } \lambda = 1, \\ \frac{1}{1-\varepsilon}\big(p_{w,\omega}(1,y,y') - \varepsilon \frac{\mathbb{1}\{y' \in B_1^{d_2}(y)\}}{\operatorname{vol}(d_2)}\big) & \text{if } \lambda = 0, \end{cases} \tag{2.19}$$

and set

$$K_{y,\omega}^{(\lambda)}(w)[O] = \int_{\mathbb{R}^{d_2}} dy' h(w,\lambda,y,y',\omega) \frac{\tilde{E}_{y,y'}[\mathcal{E}(w,\omega), O]}{\tilde{E}_{y,y'}[\mathcal{E}(w,\omega)]}. \tag{2.20}$$

In view of (2.7), this kernel is well defined. To check the measurability of the kernel, one uses a result of [22], see Theorem 44 on p. 158. The same result can also be used to show (2.16). It is then straightforward to see that the resulting kernel $K_{y,\omega}(w,\lambda_.)$ fulfills (1)–(4). □

Remark 2.3 In the notation $\mathbf{\Lambda}^{\varepsilon,\lambda}[\,\cdot\,] \stackrel{\text{def}}{=} \mathbf{\Lambda}^{\varepsilon}[\,\cdot \mid \Lambda_0 = \lambda]$ for $\lambda \in \{0,1\}$ and $K_{y,\omega}^n(w,\lambda_.) \stackrel{\text{def}}{=} K_{y,\tau_{(w(n),0)}(\omega)}(w(n+\cdot) - w(n), \lambda_.)$ for integer $n \geq 0$ and $(y,\omega,w,\lambda_.) \in \mathbb{R}^{d_2} \times \Omega \times W_0^{d_1} \times \{0,1\}^{\mathbb{Z}}$, we find, as a consequence of (2.14) and the fact that $K_{y,\omega}[X_{\cdot\wedge n}^2 \in \star]$ depends on $w([0,n]), \lambda_0, \ldots, \lambda_{n-1}$ only, that, for a fixed Brownian path $w \in W_0^{d_1}$, $\mathbf{\Lambda}^{\varepsilon} \times K_{y,\omega}$-a.s.,

$$\begin{aligned} &\mathbf{\Lambda}^{\varepsilon} \times K_{y,\omega}\big[\big(X_{n+\cdot}^2, \Lambda_{n+\cdot}\big) \in \star \,|\, \mathcal{U}_n \otimes \sigma(\Lambda_0,\ldots,\Lambda_n)\big] \\ &\quad = \mathbf{\Lambda}^{\varepsilon,\Lambda_n} \times K_{X_n^2,\omega}^n\big[\big(X_\cdot^2, \Lambda_\cdot\big) \in \star\big]. \end{aligned} \tag{2.21}$$

Remark 2.4 Thank to Girsanov's Theorem, we were able to construct the above kernels quite explicitly, so that we have a very concrete way to write expectations of X_k^2 for integers $k \geq 1$ under the quenched kernel $K_{0,\omega}$ using the formulas for the kernels for one time unit, see (2.20). Indeed, applying (2.14) successively for $n = k-1, \ldots, 1$, we find that, for all $(w,\lambda_.) \in W_0^{d_1} \times \{0,1\}^{\mathbb{Z}}$,

$$E^{K_{0,\omega}}\big[X_k^2\big] = E^{K_{0,\omega}}\big[E^{K_{X_1^2,\hat{\omega}_1} \circ \theta_1}\big[\cdots E^{K_{X_1^2,\hat{\omega}_{k-1}} \circ \theta_{k-1}}\big[X_1^2\big]\cdots\big]\big] \tag{2.22}$$

with $\hat{\omega}_i = \tau_{(w(i),0)}(\omega)$, $i = 1,\ldots,k-1$. Using identities (2.18) and (2.20) in the proof of Theorem 2.2, we obtain with $y_0 := 0$ that the right-hand side of (2.22) equals

$$\int_{\mathbb{R}^{d_2}} \cdots \int_{\mathbb{R}^{d_2}} dy_1 \cdots dy_k \prod_{i=0}^{k-1} h\big(w(i+\cdot) - w(i), \lambda_i, y_i, y_{i+1}, \hat{\omega}_i\big) y_k. \tag{2.23}$$

2.2 The Cut Times T^k

In this section, we will define the cut times which are at the heart of this work, see (2.24). The assumption $d_1 \geq 5$ becomes crucial at this point since it ensures the existence of these times, see (2.27), (2.28). For a concise review of the work of Erdös on cut times and more recent results, see, for instance, [18] and references therein. The consideration of these times is crucial to find certain decoupling effects in the process $\chi_.$ under the annealed measure $P \times K_0$, see Proposition 2.8, providing a comparison of $(\chi_k)_{k\geq 1}$ under $P \times K_0$ with an ergodic sequence. This enables us to deduce rather easily a law of large numbers.

For $r \geq 0$ and a subset A of $\mathbb{R}^{d_1}$, we define A^r as the closed r-neighborhood of A. For $(w, \lambda_.) \in W_0^{d_1} \times \{0,1\}^{\mathbb{Z}}$, we define the set of cut times as

$$\mathcal{C}(w,\lambda_.) \stackrel{\text{def}}{=} \left\{n \in \mathbb{Z} \mid \left(X^1_{(-\infty,n-1]}(w)\right)^R \cap \left(X^1_{[n,\infty)}(w)\right)^R = \emptyset,\ \Lambda_{n-1}(\lambda_.) = 1\right\} \tag{2.24}$$

and consider the point process on $\mathbb{Z}$

$$N\big((w,\lambda_.);dk\big) = \sum_{n\in\mathbb{Z}} \delta_n(dk)\mathbb{1}_{\{n\in\mathcal{C}(w,\lambda_.)\}}, \tag{2.25}$$

which is stationary for θ_1 under the measure P. It will turn out that the point process N is double infinite, i.e., the event

$$W \stackrel{\text{def}}{=} \left\{(w,\lambda_.) \in W_0^{d_1} \times \{0,1\}^{\mathbb{Z}} \,\middle|\, N\big((w,\lambda_.);\mathbb{Z}_-\big) = \infty = N\big((w,\lambda_.);\mathbb{Z}_+\big)\right\} \tag{2.26}$$

has full P-probability, see Lemma 2.6 below. We will thus restrict P, see (2.12), on the shift-invariant set W. With $\mathcal{W}$ we denote the restriction of $\mathcal{W}_0 \otimes \mathcal{S}$ to W.

Remark 2.5 On the event $\Lambda_{n-1} = 1$, $n \geq 1$, we have a very good control on the position of χ_n by the knowledge of χ_{n-1} without any further information about the environment. Due to finite range dependence, this will lead to a certain decoupling effect between the environment seen from the process $\chi_.$ after a cut time n and the environment affecting the process $\chi_.$ before time $n-1$. As a consequence, we will find the key identity in law stated in Proposition 2.8.

Lemma 2.6 $(d_1 \geq 5)$

$$P[0 \in \mathcal{C}] \geq c_1(\varepsilon) > 0, \tag{2.27}$$

$$P[W] = 1, \quad \textit{and hence on } W, \quad N\big((w,\lambda_.);dk\big) = \sum_{n\in\mathbb{Z}} \delta_{T^n(w,\lambda_.)}(dk), \tag{2.28}$$

where $T^n, n \in \mathbb{Z}$, are $\mathbb{Z}$-valued random variables on W that are increasing in n such that $T^0 \leq 0 < T^1$,

$$\hat{P} \stackrel{\text{def}}{=} P[\,\cdot\mid 0 \in \mathcal{C}] \quad \textit{is invariant under } \hat{\theta}_1 \stackrel{\text{def}}{=} \theta_{T^1}, \tag{2.29}$$

$$T^{n+m} = T^n + T^m \circ \hat{\theta}_n \quad \textit{for all } n, m \in \mathbb{Z}, \tag{2.30}$$

$$E^{\hat{P}}\big[T^1\big] = P[0 \in \mathcal{C}]^{-1}, \tag{2.31}$$

$$E^P[f] = \frac{E^{\hat{P}}[\sum_{k=0}^{T^1-1} f \circ \theta_k]}{E^{\hat{P}}[T^1]} \tag{2.32}$$

for any bounded measurable function f on W,

$$P\big[T^1 > n\big] \le c_2 (\log n)^{1+\frac{d_1-4}{2}} n^{-\frac{d_1-4}{2}}, \quad n \ge 2, \tag{2.33}$$

for $c_2(\varepsilon)$ a positive constant.

Proof Let us define, for $w \in W_0^{d_1}$,

$$B_t^1(w) \overset{\text{def}}{=} w(-t), \quad t \ge 0,$$

$$B_t^2(w) \overset{\text{def}}{=} w(t), \quad t \ge 0.$$

Noting that $B_\cdot^1, B_\cdot^2$, and $\lambda_\cdot$ are mutually independent, and $B_\cdot^1, B_\cdot^2$ are two d_1-dimensional standard Brownian motions on $(W_0^{d_1} \times \{0,1\}^{\mathbb{Z}}, \mathcal{W}_0 \otimes \mathcal{S}, P)$, we find by using the Markov property of Brownian motion that

$$P[0 \in \mathcal{C}] = \varepsilon \int_{\mathbb{R}^{d_1}} p_{d_1}(1, 0, x) P\big[\big(x + B^1_{[0,\infty)}\big)^R \cap \big(B^2_{[0,\infty)}\big)^R = \emptyset\big]\, dx.$$

To prove (2.27) it suffices to show that, for some set $A \subseteq \mathbb{R}^{d_1}$ of positive Lebesgue measure,

$$P\big[\big(x + B^1_{[0,\infty)}\big)^R \cap \big(B^2_{[0,\infty)}\big)^R = \emptyset\big] > 0 \quad \text{for all } x \in A. \tag{2.34}$$

For $i, j \ge 0$, let us define the event

$$A_{i,j} = \big\{\big(B^1_{[i,i+1]}\big)^R \cap \big(B^2_{[j,j+1]}\big)^R \neq \emptyset\big\}. \tag{2.35}$$

From the Markov property and the independence of $B_\cdot^1$ and $B_\cdot^2$ it follows for $(i, j) \neq (0, 0)$ that

$$\begin{aligned} P[A_{i,j}] &= \int_{\mathbb{R}^{d_1}} p_{d_1}(i+j, 0, x) P\big[\big(x + B^1_{[0,1]}\big)^R \cap \big(B^2_{[0,1]}\big)^R \neq \emptyset\big]\, dx \\ &\le \int_{\mathbb{R}^{d_1}} p_{d_1}(i+j, 0, x) P\Big[|x| \le \sup_{0\le s\le 1} \big|B_s^1\big| + \sup_{0\le s\le 1} \big|B_s^2\big| + 2R\Big]\, dx. \end{aligned} \tag{2.36}$$

Using Fubini and the fact that $p_{d_1}(i+j, 0, z) \le c(i+j)^{-d_1/2}$, we obtain that

$$P[A_{i,j}] \le \frac{c}{(i+j)^{d_1/2}} \Big(E^P\Big[\sup_{0\le s\le 1} \big|B_s^1\big|^{d_1}\Big] + R^{d_1}\Big) \le \frac{c}{(i+j)^{d_1/2}}, \tag{2.37}$$

which implies, since $d_1 \geq 5$,

$$\sum_{i,j=0}^{\infty} P[A_{i,j}] < \infty. \tag{2.38}$$

In analogy to the proof of Proposition 3.2.2 in [17], where intersection probabilities of two independent randome walks are investigated, we call (i,j) a *-last intersection if $A_{i,j}$ occurs while $A_{i',j'}$ for $i' \geq i$, $j' \geq j$ with $(i',j') \neq (i,j)$ do not. Because of (2.38) and Borel–Cantelli's Lemma, we know that P-a.e. the pair of paths $(B_t^1(w))_{t\geq 0}$, $(B_t^2(w))_{t\geq 0}$ has at least one such *-last intersection. Hence,

$$1 \leq \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} P\big[(i,j) \text{ is a *-last intersection}\big],$$

which implies the existence of a pair (I,J) such that

$$\begin{aligned} 0 &< P\big[(I,J) \text{ is a *-last intersection}\big] \leq P\big[\big(B^1_{[I+1,\infty)}\big)^R \cap \big(B^2_{[J+1,\infty)}\big)^R = \emptyset\big] \\ &= \int_{\mathbb{R}^{d_1}} p_{d_1}(I+J+2,0,x) P\big[\big(x+B^1_{[0,\infty)}\big)^R \cap \big(B^2_{[0,\infty)}\big)^R = \emptyset\big]\,dx, \end{aligned}$$

where in the last equality we used the Markov property and the independence of $B^1_\cdot$ and $B^2_\cdot$. Since the integrand is nonnegative, this proves (2.34) and hence (2.27). By an analogous result for simple stationary point processes on $\mathbb{Z}$ as Lemma II.12 in [20], one finds using the ergodicity of θ_1 that (2.28) holds true. The measure $\hat{P}$ corresponds up to a multiplicative constant to the Palm measure attached to the stationary point process N, see Chap. II in [20], in particular, (10) on p. 317. The statements (2.29)–(2.32) are then standard consequences. Note that (2.32) is a consequence of (19) on p. 331 of [20] and that (2.31) follows from (2.32) with the choice $f = \mathbb{1}_{\{0\in\mathcal{C}\}}$. It remains to show (2.33). For integer $L \geq 1$ and for $j \geq 0$, we define

$$k_j := 1 + Lj.$$

For $J \geq 1$, we find that

$$\begin{aligned} P\big[T^1 > k_{3J}\big] &= P\big[N\big((w,\lambda_\cdot);[1,k_{3J}]\big) = 0\big] \\ &\leq P\Bigg[N\big((w,\lambda_\cdot);[1,k_{3J}]\big) = 0, \\ &\qquad \bigcap_{j=0}^{3J}\big\{\big(X^1_{(-\infty,k_j-1]}\big)^R \cap \big(X^1_{[k_j+1,\infty)}\big)^R = \emptyset\big\}\Bigg] \\ &\quad + \sum_{j=0}^{3J} P\big[\big(X^1_{(-\infty,k_j-1]}\big)^R \cap \big(X^1_{[k_j+1,\infty)}\big)^R \neq \emptyset\big] \\ &=: a_1 + a_2. \end{aligned}$$

First we bound a_2. Note that, for integer $n \geq 1$,

$$P\big[\big(B^1_{[0,\infty)}\big)^R \cap \big(B^2_{[n,\infty)}\big)^R \neq \emptyset\big] \leq \sum_{i\geq 0, j\geq n} P[A_{i,j}] \overset{(2.37)}{\leq} c\sum_{j\geq n} j^{1-\frac{d_1}{2}} \leq cn^{-\frac{d_1-4}{2}}, \tag{2.39}$$

and hence, by the stationarity of Brownian motion,

$$a_2 = (3J+1)P\big[\big(B^1_{[0,\infty)}\big)^R \cap \big(B^2_{[L+1,\infty)}\big)^R \neq \emptyset\big] \leq c(3J+1)(L+1)^{-\frac{d_1-4}{2}}. \tag{2.40}$$

Now we turn to the control of a_1. For $j = 1, \ldots, 3J$, observe that, on the event $\{N((w, \lambda_.); [1, k_{3J}]) = 0)\}$, the following inclusion holds:

$$\begin{aligned}&\big\{\big(X^1_{(-\infty,k_{j-1}-1]}\big)^R \cap \big(X^1_{[k_j,\infty)}\big)^R = \emptyset\big\} \cap \big\{\big(X^1_{(-\infty,k_j-1]}\big)^R \cap \big(X^1_{[k_{j+1},\infty)}\big)^R = \emptyset\big\}\\ &\quad\subseteq \big\{\big(X^1_{[k_{j-1}-1,k_j-1]}\big)^R \cap \big(X^1_{[k_j,k_{j+1}]}\big)^R \neq \emptyset\big\} \cup \{\lambda_{k_j-1} = 0\}.\end{aligned}$$

We thus find that the event

$$\bigcap_{j=3,6,\ldots}^{3J} \big\{\big(X^1_{[k_{j-1}-1,k_j-1]}\big)^R \cap \big(X^1_{[k_j,k_{j+1}]}\big)^R \neq \emptyset\big\} \cup \{\lambda_{k_j-1} = 0\}$$

occurs, whenever the event considered in a_1 occurs. By the independence of Brownian increments and the fact that θ_1 preserves P we obtain that

$$a_1 \leq P\big[\big\{\big(X^1_{[0,L]}\big)^R \cap \big(X^1_{[L+1,2L+1]}\big)^R \neq \emptyset\big\} \cup \{\lambda_L = 0\}\big]^J \leq P[0 \notin \mathcal{C}]^J. \tag{2.41}$$

Choosing a large enough γ which depends on d_1, R, and ε and setting $J = [\gamma \log n]$, $L = [\frac{n}{3J}]$, we obtain (2.33) from (2.40) and (2.41). □

2.3 A Decoupling Effect and a Law of Large Numbers

Now we will exploit the presence of cut times, see (2.28), in order to produce decoupling in the process $\chi_.$ under the measure $P \times K_0$, see (2.17). For this purpose, we introduce the process $Z_.$ living on an enlarged space, see below (2.42), equipped with a measure Q^0, that uses our previous coupling construction and the cut times, see (2.43) and (2.44). The idea behind the construction of the process $Z_.$ is to start after each cut time a fresh path for $X^2_.$ in a new environment, which is chosen independently from the previous environment, see Remark 2.5. We then recover the law of the process $\chi_.$ at integer times under $P \times K_0$, see Proposition 2.8.

First, we have to introduce some further notation. Consider the product spaces

$$\Gamma^0 \overset{\text{def}}{=} W \times \big(W^{d_2}_+ \times \Omega\big)^{\mathbb{N}}, \qquad \Gamma^s \overset{\text{def}}{=} W \times \big(W^{d_2}_+ \times \Omega\big)^{\mathbb{Z}} \tag{2.42}$$

endowed with their product σ-algebras, see (2.26) for the definition of W. Recall at this point the definition of $\hat{P}$, see (2.29), and note that in the sequel all the measures denoted with a ˆ correspond up to a different normalization to the Palm measure attached to the point process $N((w,\lambda_.);dk)$, see (2.25). On the spaces defined in (2.42), we introduce the measures

$$Q^0 \stackrel{\text{def}}{=} P \times M^0, \qquad \hat{Q}^0 \stackrel{\text{def}}{=} \hat{P} \times M^0, \qquad Q^s \stackrel{\text{def}}{=} P \times M^s, \qquad \hat{Q}^s \stackrel{\text{def}}{=} \hat{P} \times M^s, \tag{2.43}$$

where M^0 and M^s stand for the kernels from W to $(W_+^{d_2} \times \Omega)^{\mathbb{N}}$ respectively from W to $(W_+^{d_2} \times \Omega)^{\mathbb{Z}}$ defined by

$$M^0\big((w,\lambda_.);d\gamma^0\big) = K_0\big((w,\lambda_.);du_0\,d\omega_0\big) \otimes \bigotimes_{m\geq 1} K_0\big(\theta_{T^m}(w,\lambda_.);du_m\,d\omega_m\big), \tag{2.44}$$

recalling the definition (2.17), with $\gamma^0 = (u_m,\omega_m)_{m\geq 0} \in (W_+^{d_2} \times \Omega)^{\mathbb{N}}$, and similarly

$$M^s\big((w,\lambda_.);d\gamma^s\big) = \bigotimes_{m\in\mathbb{Z}} K_0\big(\theta_{T^m}(w,\lambda_.);du_m\,d\omega_m\big) \tag{2.45}$$

with $\gamma^s = (u_m,\omega_m)_{m\in\mathbb{Z}} \in (W_+^{d_2} \times \Omega)^{\mathbb{Z}}$. On Γ^0, we define the process $(Z_t)_{t\geq 0}$ by

$$Z_t \stackrel{\text{def}}{=} \big(X_t^1, Y_t\big), \quad t \geq 0 \tag{2.46}$$

with $X_.^1$ defined in (2.3) and

$$\begin{aligned} Y_t &\stackrel{\text{def}}{=} u_0(t) \quad \text{for } 0 \leq t < T^1, \quad \text{and} \\ Y_{(T^m+t)\wedge T^{m+1}} &\stackrel{\text{def}}{=} Y_{T^m} + u_m\big(t \wedge \big(T^{m+1} - T^m\big)\big) \quad \text{for } m \geq 1,\ t \geq 0. \end{aligned} \tag{2.47}$$

Note that $Z_0 = 0$, Q^0-a.s. Loosely speaking, the process $Z_.$ is constructed by attaching after each cut time a new path for the $\mathbb{R}^{d_2}$-components which evolves in a new independent environment. Similarly we define the two-sided process $(Z_t^s)_{t\in\mathbb{R}}$ on Γ^s by

$$Z_t^s \stackrel{\text{def}}{=} \big(X_t^1, Y_t^s\big), \quad t \in \mathbb{R}, \tag{2.48}$$

where, for $m \in \mathbb{Z}$ and $t \in \mathbb{R}_+$,

$$\begin{aligned} Y_0^s &\stackrel{\text{def}}{=} 0, \\ Y^s_{(T^m+t)\wedge T^{m+1}} &\stackrel{\text{def}}{=} Y^s_{T^m} + u_m\big(t \wedge \big(T^{m+1} - T^m\big)\big), \end{aligned} \tag{2.49}$$

and we introduce also the Ω-valued process $(\alpha_t^s)_{t\in\mathbb{R}}$ by

$$\alpha_t^s \stackrel{\text{def}}{=} \tau_{Z_t^s - Z^s_{T^m}}(\omega_m) \quad \text{for } T^m \leq t < T^{m+1},\ m \in \mathbb{Z}, \tag{2.50}$$

which plays the role of the "relevant environment viewed from the particle." Note that, by definition, $Z_0^s = 0$, Q^s-a.s.

Remark 2.7 Note that by definition we have that, under the measure $\hat{Q}^s$, the joint distribution of T^1 and the piece of trajectory $(Z^s_t)_{t\in[0,T^1]}$ is the same as the joint distribution of T^1 and $(\chi_t)_{t\in[0,T^1]}$ under $\hat{P}\times K_0$, see (2.3), (2.4) for the definition of $\chi_.$ and recall that $\hat{P}[T^0=0]=1$.

The following proposition yields a crucial identity in law.

Proposition 2.8 *Under the measure Q^0, the sequence of random vectors $(Z_n)_{n\geq 0}$ has the same law as $(\chi_n)_{n\geq 0}$ under the measure $P\times K_0$.*

Proof The idea of the proof is to fix $(w,\lambda_.)\in W$ and then to show by induction that for all integers $m\geq 0$ the following statement holds:

$$\text{For all bounded measurable functions } f^k,\ k=0,\dots,m,\ \text{on } \mathbb{R}^d,$$
$$E^{K_0}\left[\prod_{k=0}^{m} f^k(\chi_k)\right]=E^{M^0}\left[\prod_{k=0}^{m} f^k(Z_k)\right].\tag{2.51}$$

Proposition (2.8) then follows by integrating out with respect to P, see (2.43) for the definition of Q^0. Let us fix $(w,\lambda_.)\in W$ and note that (2.51) holds true for $0\leq m\leq T^1(w,\lambda_.)$ by definition, see (2.44), (2.46), and (2.47). We assume the above statement to be true for m and show that it must still hold for $m+1$. Without loss of generality we can assume that $l=T^N<m+1\leq T^{N+1}$ for an integer $1\leq N\leq m$. Recall that $K_0=\mathbb{P}\times K_{0,\omega}$, see (2.17), and so, applying (2.14) with $n=l$ and then with $n=l-1$, we obtain that

$$\begin{aligned}E^{K_0}\left[\prod_{k=0}^{m+1} f^k(\chi_k)\right]&=\mathbb{E}\times E^{K_{0,\omega}}\left[\prod_{k=0}^{l-1} f^k(\chi_k)E^{K_{X^2_{l-1},\hat{\omega}}\circ\theta_{l-1}}\left[f^l\big(w(l),X^2_1\big)\right.\right.\\&\qquad\left.\left.\times E^{K_{X^2_1,\tilde{\omega}}\circ\theta_l}\left[\prod_{k=1}^{m+1-l} f^{l+k}\big(w(l+k),X^2_k\big)\right]\right]\right]\end{aligned}\tag{2.52}$$

with $\hat{\omega}=\tau_{(w(l-1),0)}(\omega)$ and $\tilde{\omega}=\tau_{(w(l),0)}(\omega)$. Since $l=T^N$ is a cut time, we have that $\lambda_{l-1}=1$, see (2.24), and hence with (3) of Theorem 2.2 and (2.15) we find that the right-hand side of (2.52) is equal to

$$\begin{aligned}&\int_{\mathbb{R}^{d_2}}\frac{dy}{\mathrm{vol}(d_2)}\mathbb{E}\left[E^{K_{0,\omega}}\left[\prod_{k=0}^{l-1} f^k(\chi_k)\mathbb{1}_{\{y\in B^{d_2}_1(X^2_{l-1})\}}\right]f^l\big(w(l),y\big)\right.\\&\qquad\left.\times E^{K_{0,\bar{\omega}}\circ\theta_l}\left[\prod_{k=1}^{m+1-l} f^{l+k}\big(w(l+k),y+X^2_k\big)\right]\right]\end{aligned}\tag{2.53}$$

with $\bar{\omega}=\tau_{(w(l),y)}(\omega)$, where we also used Fubini's Theorem. From the definition of the cut times T^k, see (2.24) and Lemma 2.6, and the measurability property (2.16) we see that all the factors under the $\mathbb{P}$-expectation in (2.53) are independent, see (1.5). Together with the induction hypothesis and stationarity of the environment (i.e., $\tau_x\mathbb{P}=\mathbb{P}$), we obtain that (2.53) is equal to

$$\int_{\mathbb{R}^{d_2}} \frac{dy}{\mathrm{vol}(d_2)} E^{M^0}\left[\prod_{k=0}^{l-1} f^k(Z_k)\mathbb{1}_{\{y\in B_1^{d_2}(Y_{l-1})\}}\right] f^l\big(w(l),y\big)$$

$$\times E^{K_0\circ\theta_l}\left[\prod_{k=1}^{m+1-l} f^{l+k}\big(w(l+k),y+X_k^2\big)\right]. \tag{2.54}$$

Recalling the definitions (2.44), (2.46), and (2.47), we deduce with the help of Fubini's Theorem that (2.54) equals

$$E^{M^0}\left[\prod_{k=0}^{l} f^k(Z_k)E^{K_0\circ\theta_l}\left[\prod_{k=1}^{m+1-l} f^{l+k}\big(w(l+k),Y_l+X_k^2\big)\right]\right]=E^{M^0}\left[\prod_{k=0}^{m+1} f^k(Z_k)\right],$$

where we used that $\theta_l=\theta_{T^N}$. This finishes the induction step. □

Remark 2.9 By construction of the probability kernel $K_{y,\omega}(w,\lambda_\cdot)$, $y\in\mathbb{R}^d$, $\omega\in\Omega$, $(w,\lambda_\cdot)\in W$, see in particular (3) of Theorem 2.2, we have that due to $\lambda_{T^k-1}=1, k\ge 1$, see the definition of cut times (2.24), the transition from $X^2_{T^k-1}$ to $X^2_{T^k}$ depends only on the position $X^2_{T^k-1}$ without any additional information on the environment. However, the piece of trajectory X_t^2, $T^k-1\le t\le T^k$, is influenced by the environment, see (2.16). That is the reason why a decoupling effect concerning the environment, as described by the process $Z_t, t\ge 0$, under Q^0, can only be observed in the original process $\chi_t, t\ge 0$, under $P\times K_0$ when we ignore the piece of trajectory during one unit of time just before each cut time. In fact, it can be shown by the same arguments as in the proof of Lemma 2.8 that, under $P\times K_0$, the sequence of random variables $\chi_{(T^m+\cdot)\wedge(T^{m+1}-1)}, m\ge 0$, has the same law as $Z_{(T^m+\cdot)\wedge(T^{m+1}-1)}, m\ge 0$, under Q^0, where we set $T^0=0$.

We now introduce on Γ^s a shift $(\Theta_k)_{k\in\mathbb{Z}}$ via:

$$\Theta_k\big((w,\lambda_\cdot),\gamma^s\big)=\big(\theta_k(w,\lambda_\cdot),(u_{m+n},\omega_{m+n})_{m\in\mathbb{Z}}\big)$$

$$\text{on } T^n(w,\lambda_\cdot)\le k<T^{n+1}(w,\lambda_\cdot) \tag{2.55}$$

with $\gamma^s=(u_m,\omega_m)_{m\in\mathbb{Z}}\in(W_+^{d_2}\times\Omega)^{\mathbb{Z}}$.

Proposition 2.10 *For all $\bar{\gamma^s}\in\Gamma^s$, the following identities hold*:

$$Z^s_{l+a}\big(\bar{\gamma^s}\big)-Z^s_l\big(\bar{\gamma^s}\big)=Z^s_a\circ\Theta_l\big(\bar{\gamma^s}\big)\quad \textit{for } l\in\mathbb{Z},\ a\in\mathbb{R}_+, \tag{2.56}$$

$$Z^s_n\big(\bar{\gamma^s}\big)=\sum_{k=0}^{n-1} Z^s_1\circ\Theta_k\big(\bar{\gamma^s}\big)\quad \textit{for integers } n\ge 1, \tag{2.57}$$

$$\alpha^s_u\big(\bar{\gamma^s}\big)=\alpha^s_{u_r}\circ\Theta_{[u]}\big(\bar{\gamma^s}\big)\quad \textit{for } u=[u]+u_r\in\mathbb{R}. \tag{2.58}$$

Moreover,

$$\Theta_1 \text{ preserves } Q^s \text{ and in fact } \big(\Gamma^s, \Theta_1, Q^s\big) \text{ is ergodic,} \tag{2.59}$$

$$E^{Q^s}[f] = \frac{E^{\hat{Q}^s}\big[\sum_{k=0}^{T^1-1} f\circ\Theta_k\big]}{E^{\hat{P}}[T^1]} \tag{2.60}$$

for any bounded measurable function f on Γ^s,

$$Z_1^s \in L^m\big(Q^s\big) \quad \text{for all } m \in [1,\infty) \text{ when } d_1 \geq 5. \tag{2.61}$$

Proof Identities (2.56)–(2.58) follow by direct inspection of the definitions (2.48)–(2.50) and (2.55). The proof of (2.59) exactly follows the proof in the discrete setting (see [4], pp. 534–535). There are slight differences in the notation. The objects Γ_s, Q_s, M_s, $\{\tilde{w}_m\}_{m\in\mathbb{Z}}$, and $\mathcal{D}$ in [4] correspond to our Γ^s, Q^s, M^s, $\{u_m\}_{m\in\mathbb{Z}}$, and $\mathcal{C}$. Further, one has to read $(w,\lambda_.)$ instead of w in the proof in [4]. Let us point out that the main strategy in showing the ergodicity of (Γ^s,Θ_1,Q^s) is to prove that $(\Gamma^s\cap\{0\in\mathcal{C}\}, \hat{\Theta}_1 \stackrel{\text{def}}{=} \Theta_{T^1}, \hat{Q}^s)$ is ergodic, which is indeed an equivalent statement, see (34) on p. 357 in [20]. Analogously to (2.32), we find (2.60) as a standard consequence of the first part of the statement in (2.59). We now come to the proof of (2.60). We choose $m\in[1,\infty)$, then by definition (2.48),

$$E^{Q^s}\big[\big|Z_1^s\big|^m\big] \leq 2^{m-1}\big\{E^{Q^s}\big[\big|X_1^1\big|^m\big] + E^{Q^s}\big[\big|Y_1^s\big|^m\big]\big\}. \tag{2.62}$$

The first expectation on the right-hand side of (2.62) is finite, since $X^1_.$ is a standard d_1-dimensional Brownian motion under Q^s. In the notation (2.43), (2.45), and (2.49), we have that

$$\begin{aligned}
E^{Q^s}\big[\big|Y_1^s\big|^m\big] &= E^P\big[E^{K_0\circ\theta_{T^0}}\big[\big|u_0\big(1-T^0\big)-u_0\big(-T^0\big)\big|^m\big]\big] \\
&= \sum_{n\geq 0} E^P\big[T^0=-n,\ E^{K_0\circ\theta_{-n}}\big[\big|u_0(1+n)-u_0(n)\big|^m\big]\big] \\
&\stackrel{\text{stat.}}{=} \sum_{n\geq 0} E^{\hat{P}}\big[T^1>n,\ E^{K_0}\big[\big|u_0(1+n)-u_0(n)\big|^m\big]\big]P\big[T^0=0\big].
\end{aligned}$$

If we show that the above expectation with respect to the measure K_0 is uniformly bounded, then (2.60) follows, since $\sum_{n\geq 0}\hat{P}[T^1>n] = E^{\hat{P}}[T^1] = P[T^0=0]^{-1} < \infty$, see (2.31) and (2.27). Indeed, by construction of the kernel $K_0 = \mathbb{P}\times K_{0,\omega}$, see (2.17)–(2.20), we find that, for each fixed $(w,\lambda_.)\in W$,

$$\begin{aligned}
&E^{K_0}\big[\big|u_0(1+n)-u_0(n)\big|^m\big] \\
&\quad = \mathbb{E}\bigg[E^{K_{0,\omega}}\bigg[\int_{\mathbb{R}^{d_2}} h\big(w(n+\cdot)-w(n),\lambda_n,u_0(n),y,\hat{\omega}\big)\big|y-u_0(n)\big|^m\,dy\bigg]\bigg]
\end{aligned} \tag{2.63}$$

with $\hat{\omega} = \tau_{(w(n),0)}(\omega)$. When $\lambda_n = 1$, we immediately see by the definition of h, see (2.19), that the integral under the expectation is $K_{0,\omega}$-a.s. bounded by 1. In the other case, when $\lambda_n = 0$, the above integral is $K_{0,\omega}$-a.s. less than or equal to

$$\frac{1}{1-\varepsilon}\int_{\mathbb{R}^{d_2}} p_{w(n+\cdot)-w(n),\hat{\omega}}\big(1, u_0(n), y\big)\big|y-u_0(n)\big|^m\,dy + \frac{\varepsilon}{1-\varepsilon}. \tag{2.64}$$

A result in [12] concerning exponential bounds on fundamental solutions of parabolic equations of second order, see Theorem 1 on p. 67, tells us that

$$p_{w(n+\cdot)-w(n),\hat{\omega}}\big(1, u_0(n), y\big) \le c_3(w,\omega)e^{-c_4(w,\omega)|y-u_0(n)|^2} \tag{2.65}$$

for some positive constants $c_3(w,\omega)$, $c_4(w,\omega)$. A closer look into the proof of the applied result from [12] reveals that the constants c_3 and c_4 in (2.65) can indeed be chosen to be independent of the Brownian path w and the environment ω due to the uniform boundedness and Lipschitz constant of the drift b, see (1.3). With this in mind, combining (2.65) and (2.64), one easily sees that (2.63) is also uniformly bounded in the case where $\lambda_n = 0$. This finishes the proof of (2.60). □

Now we are ready to state a law of large numbers when $d_1 \ge 5$. For the notation, see (2.17), (2.29), (2.43), (2.45), (2.48)–(2.50).

Theorem 2.11 ($d_1 \ge 5$)

P_0-a.s.,

$$\frac{X_t}{t} \underset{t\to\infty}{\longrightarrow} v \stackrel{\text{def}}{=} \frac{E^{\hat{P}\times K_0}[\int_0^{T^1} b(\chi_u,\omega)\,du]}{E^{\hat{P}}[T^1]} = E^{Q^s}\left[\int_0^1 b\big(\alpha_u^s\big)\,du\right] \tag{2.66}$$
$$= E^{Q^s}\big[Z_1^s\big].$$

Proof First, we prove that

$$P_0\text{-a.s.,}\qquad \lim_{t\to\infty}\frac{X_t}{t} = E^{Q^s}\big[Z_1^s\big]. \tag{2.67}$$

For all $t \ge 1$,

$$\left|\frac{X_t}{t} - E^{Q^s}\big[Z_1^s\big]\right| \le \frac{1}{t}|X_t - X_{[t]}| + \left|\frac{X_{[t]}}{[t]}\cdot\frac{[t]}{t} - E^{Q^s}\big[Z_1^s\big]\right|. \tag{2.68}$$

For $\omega \in \Omega$, under $P_{0,\omega}$, the process $(W_t')_{t\ge 0}$ defined as

$$W_t' \stackrel{\text{def}}{=} X_t - X_0 - \int_0^t b(X_s,\omega)\,ds$$

is a d-dimensional Brownian motion and, $P_{0,\omega}$-a.s.,

$$\frac{1}{t}|X_t - X_{[t]}| = \frac{1}{t}\left|\int_{[t]}^t b(X_s,\omega)\,ds + \int_{[t]}^t dW_s'\right|$$
$$\le \frac{1}{t}\big(\kappa + \big|W_t' - W_{[t]}'\big|\big). \tag{2.69}$$

A standard application of Borel–Cantelli's Lemma and Bernstein's inequality shows that the last expression in (2.69) converges $P_{0,\omega}$-a.s. to 0 as $t \to \infty$. Together with (2.68), we see that to prove (2.67) it suffices to show for integers $n \geq 1$ that, P_0-a.s., $\frac{1}{n} X_n$ converges to $E^{Q^s}[Z_1^s]$ as $n \to \infty$. As a consequence of (1) of Theorem 2.2 and Proposition 2.8, we therefore obtain (2.67), once we show that $\frac{Z_n}{n} \to E^{Q^s}[Z_1^s]$, Q^0-a.s., as $n \to \infty$. As we will now see, the latter claim follows from the convergence of $\frac{Z_n^s}{n}$ under Q^s, which is an immediate consequence of (2.57), (2.59), (2.60), and Birkhoff's Ergodic Theorem. Indeed, we construct an enlarged probability space on which both processes $Z_.$ and $Z_.^s$ can be defined. Consider the product space

$$\Gamma \stackrel{\text{def}}{=} W \times \big(W_+^{d_2} \times \Omega\big) \times \big(W_+^{d_2} \times \Omega\big)^{\mathbb{Z}} \tag{2.70}$$

endowed with its product σ-algebra and the measure

$$Q \stackrel{\text{def}}{=} P \times M, \tag{2.71}$$

where M is the probability kernel from W to $(W_+^{d_2} \times \Omega) \times (W_+^{d_2} \times \Omega)^{\mathbb{Z}}$ defined as

$$M\big((w,\lambda_.); d\gamma\big) = K_0\big((w,\lambda_.); du_0'\, d\omega_0'\big) \otimes \bigotimes_{m\in\mathbb{Z}} K_0\big(\theta_{T^m}(w,\lambda_.); du_m\, d\omega_m\big) \tag{2.72}$$

with $\gamma = ((u_0', \omega_0'), (u_m, \omega_m)_{m\in\mathbb{Z}}) \in (W_+^{d_2} \times \Omega) \times (W_+^{d_2} \times \Omega)^{\mathbb{Z}}$. With the projections

$$\begin{aligned} &\pi^0 : \big((w,\lambda_.),\gamma\big) \in \Gamma \longmapsto \big((w,\lambda_.), (u_0', \omega_0'), (u_m, \omega_m)_{m\geq 1}\big) \in \Gamma^0, \\ &\pi^s : \big((w,\lambda_.),\gamma\big) \in \Gamma \longmapsto \big((w,\lambda_.), (u_m, \omega_m)_{m\in\mathbb{Z}}\big) \in \Gamma^s, \end{aligned} \tag{2.73}$$

we find that $Q^0 = \pi^0 \circ Q$ and $Q^s = \pi^s \circ Q$. We thus obtain that, under Q, the processes

$$\tilde{Z}_t \stackrel{\text{def}}{=} Z_t \circ \pi^0, \quad t \geq 0, \quad \text{and} \quad \tilde{Z}_t^s \stackrel{\text{def}}{=} Z_t^s \circ \pi^s, \quad t \in \mathbb{R}, \tag{2.74}$$

defined on Γ have the same law as our original processes $Z_.$ and $Z_.^s$ under Q^0 and Q^s respectively. Since, Q-a.s.,

$$\tilde{Z}_{T^1+t} - \tilde{Z}_{T^1} = \tilde{Z}_{T^1+t}^s - \tilde{Z}_{T^1}^s \quad \text{for all } t \geq 0, \tag{2.75}$$

it follows that, Q-a.s.,

$$\frac{1}{t}\big|\tilde{Z}_t - \tilde{Z}_t^s\big| \leq \frac{1}{t} \sup_{a\in[0,T^1]} \big|\tilde{Z}_a - \tilde{Z}_a^s\big| \xrightarrow{t\to\infty} 0. \tag{2.76}$$

We thus find that $\frac{\tilde{Z}_n^s}{n}$ and $\frac{\tilde{Z}_n}{n}$ have the same limit Q-a.s., which concludes the proof of (2.67). We now show the second and the third equality in (2.66). First, we show that

$$\lim_{n\to\infty} \frac{E^{\hat{P}\times K_0}[\int_0^{T^n} b(\chi_s,\omega)\, ds]}{E^{\hat{P}}[T^n]} = E^{Q^s}\big[Z_1^s\big] \tag{2.77}$$

holds, and then we find that the sequence on the left is in fact constant and equals v. Since the measure $\hat{P} \times K_0$ is absolutely continuous with respect to $P \times K_0$, it follows from (2.67) by using (1) of Theorem 2.2 and the fact that, $P \times K_0$-a.s, $W_t/t \to 0$ as $t \to \infty$,

$$\hat{P} \times K_0\text{-a.s.}, \qquad \frac{1}{t}\int_0^t b(\chi_s, \omega)\, ds \overset{t\to\infty}{\longrightarrow} E^{Q^s}\big[Z_1^s\big].$$

By dominated convergence this limit holds true in $L^1(\hat{P} \times K_0)$ as well. Because of the ergodicity of $(W \cap \{0 \in \mathcal{C}\}, \hat{\theta}_1, \hat{P})$, which is a consequence of the ergodicity of (W, θ_1, P), see (34) on p. 357 in [20], we have:

$$\frac{T^n}{n} \overset{(2.30)}{=} \frac{1}{n}\sum_{k=0}^{n-1} T^1 \circ \hat{\theta}_k \overset{n\to\infty}{\longrightarrow} E^{\hat{P}}\big[T^1\big] \overset{(2.27),(2.31)}{<} \infty, \quad \hat{P}\text{-a.s. and in } L^1(\hat{P}), \tag{2.78}$$

and we find that, $\hat{P} \times K_0$-a.s. and in $L^1(\hat{P} \times K_0)$,

$$\lim_{n\to\infty} \frac{1}{n}\int_0^{T^n} b(\chi_s, \omega)\, ds = \lim_{n\to\infty} \frac{T^n}{n}\, \frac{1}{T^n}\int_0^{T^n} b(\chi_s, \omega)\, ds = E^{Q^s}\big[Z_1^s\big]E^{\hat{P}}\big[T^1\big].$$

Together with (2.30) and (2.29), (2.77) now follows. For a fixed $(w, \lambda_\cdot) \in W \cap \{0 \in \mathcal{C}\}$ and $k \geq 1$, we find by an application of (2.14) with $n = T^k$ and then with $n = T^k - 1$ and similar considerations to those leading to (2.53) that

$$\begin{aligned}
&E^{K_0}\bigg[\int_{T^k}^{T^{k+1}} b(\chi_u, \omega)\, du\bigg] \\
&\quad = E^{K_0}\bigg[\int_0^{T^1\circ\hat{\theta}_k} b(\chi_{T^k+u}, \omega)\, du\bigg] \\
&\quad = \int_{\mathbb{R}^{d_2}} \frac{dy}{\mathrm{vol}(d_2)}\mathbb{E}\bigg[E^{K_{0,\omega}}\Big[\mathbb{1}_{\{y \in B_1^{d_2}(X^2_{T^k-1})\}}\Big] \\
&\qquad \times E^{K_{0,\bar{\omega}}\circ\hat{\theta}_k}\bigg[\int_0^{T^1\circ\hat{\theta}_k} b\big(\big(X_u^1 \circ \hat{\theta}_k, X_u^2\big), \bar{\omega}\big)\, du\bigg]\bigg]
\end{aligned}$$

with $\bar{\omega} = \tau_{(w(T^k),y)}(\omega)$. By an independence argument as above (2.54) and stationarity of the environment we finally obtain that

$$E^{K_0}\bigg[\int_{T^k}^{T^{k+1}} b(\chi_u, \omega)\, du\bigg] = E^{K_0}\bigg[\int_0^{T^1} b(\chi_u, \omega)\, du\bigg] \circ \hat{\theta}_k. \tag{2.79}$$

Recalling that the measure $\hat{P}$ is invariant under $\hat{\theta}_k$, see (2.29), we thus find

$$\begin{aligned}
E^{\hat{P}\times K_0}\bigg[\int_0^{T^n} b(\chi_u, \omega)\, du\bigg] &= \sum_{k=0}^{n-1} E^{\hat{P}\times K_0}\bigg[\int_{T^k}^{T^{k+1}} b(\chi_u, \omega)\, du\bigg] \\
&\overset{(2.29),(2.79)}{=} nE^{\hat{P}\times K_0}\bigg[\int_0^{T^1} b(\chi_u, \omega)\, du\bigg]
\end{aligned}$$

and

$$E^{\hat{P}}\big[T^n\big] \overset{(2.30)}{=} E^{\hat{P}}\left[\sum_{k=0}^{n} T^1 \circ \hat{\theta}_k\right] \overset{(2.29)}{=} nE^{\hat{P}}\big[T^1\big],$$

which shows that the sequence in (2.77) is indeed constant and equal to

$$v \overset{\text{def}}{=} \frac{E^{\hat{P}\times K_0}[\int_0^{T^1} b(\chi_u,\omega)\,du]}{E^{\hat{P}}[T^1]} = \frac{E^{\hat{Q}^s}[\int_0^{T^1} b(Z_u^s,\omega)\,du]}{E^{\hat{P}}[T^1]} = \frac{E^{\hat{Q}^s}[\int_0^{T^1} b(\alpha_u^s)\,du]}{E^{\hat{P}}[T^1]}, \tag{2.80}$$

where we used Remark 2.7 in the first equality and definition (2.50) together with the fact that $Z^s_{T^0} = Z^s_0 = 0$, $\hat{Q}^s$-a.s., in the second equality in (2.80). The second and the third equality in (2.66) then follow from (2.80) by applying (2.58) and (2.60) to the last expression in (2.80). □

Remark 2.12 The formula for the limiting velocity, see (2.66), is reasonably explicit and depends only on a finite piece of trajectory up to the first cut time after time 0 and its first moment.

3 Two Invariance Principles under the Annealed Measure

In this section we provide two central limit theorems under the annealed measure. The first one is shown under a symmetry assumption on the drift and $d_1 \geq 7$, see Theorem 3.3, whereas, for the second theorem, there is no symmetry assumption, but we need to assume that $d_1 \geq 13$.

For integer $n \geq 1$, we denote by I_n the $n \times n$-dimensional identity matrix. We further introduce the reflection

$$\mathcal{R} : \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \longmapsto \mathbb{R}^{d_1} \times \mathbb{R}^{d_2},$$

$$(x, y) \longmapsto (x, -y).$$

For the first central limit theorem, we assume the following antipodal symmetry in the last d_2 components of the drift under the measure $\mathbb{P}$:

$$\big(\mathcal{R}\big(b(z,\omega)\big)\big)_{z\in\mathbb{R}^d} \text{ has the same law as } \big(b\big(\mathcal{R}(z),\omega\big)\big)_{z\in\mathbb{R}^d}. \tag{3.1}$$

Since the first d_1 components of the drift $b(\cdot,\cdot)$ vanish, we have that $\mathcal{R}(b(\cdot,\cdot))$ equals $-b(\cdot,\cdot)$. Note that when (3.1) holds, then $\mathcal{R}(X_.)$ has the same law under P_0 as $X_.$, and $E_0[X_t] = 0$ for all $t \geq 0$. By the definition of $(W'_t)_{t\geq 0}$, see below (2.68), we have that $P_{0,\omega}$-a.s., $X_t = X_0 + \int_0^t b(X_s,\omega)\,ds + W'_t$ for each $\omega \in \Omega$. The strong law of large numbers for Brownian motion, see Problem 9.3 in [13], and Theorem 2.11 imply that P_0-a.s., $\frac{1}{t}\int_0^t b(X_s,\omega)\,ds \to v$, and hence with dominated convergence the convergence holds in $L^1(P_0)$ as well. So $E_0[X_t] = E_0[\int_0^t b(X_s,\omega)\,ds] = 0$, and we deduce that the limiting velocity in (2.66) vanishes under the assumption (3.1).

Remark 3.1 A possible example of a drift $b^*(z,\omega)$ with $z \in \mathbb{R}^d, \omega \in \Omega$, see (1.1), such that (3.1) is satisfied can be constructed as follows. We consider a canonical Poisson point process on $\mathbb{R}^d$ with constant intensity as the random environment. Pick an $\mathbb{R}^{d_2}$-valued measurable function $\varphi(z), z \in \mathbb{R}^d$, which is supported in a ball of radius $R/4$ and such that $\varphi(\mathcal{R}(z)) = -\varphi(z)$ holds for all $z \in \mathbb{R}^d$. Then make the convolution of the Poisson point process with the function φ and truncate the new function. After smoothing out with a Lipschitz continuous real-valued mollifier $\rho(z), z \in \mathbb{R}^d$, supported in a ball of radius $R/4$ and such that $\rho(\mathcal{R}(z)) = \rho(z)$ for all $z \in \mathbb{R}^d$, one obtains an example of a possible $b^*(z,\omega)$.

For two $C(\mathbb{R}_+, \mathbb{R}^d)$-valued sequences $\xi^n_\cdot$ and $\zeta^n_\cdot$, $n \geq 1$, respectively defined on the probability spaces $(\Xi_1, \mathcal{D}_1, \mu_1)$ and $(\Xi_2, \mathcal{D}_2, \mu_2)$, we say that $(\xi^n_\cdot)_{n\geq 1}$ under μ_1 is weak convergence equivalent (abbreviated by wce) to $(\zeta^n_\cdot)_{n\geq 1}$ under μ_2 if the weak convergence of the law of $\xi^n_\cdot$ under μ_1 is equivalent to the weak convergence of the law of $\zeta^n_\cdot$ under μ_2, and if both limits are the same when the weak convergence holds true.

Before we come to the main results of this section, we briefly discuss some integrability properties stated in the following:

Lemma 3.2

$$\text{For all } \eta \geq 1: \; T^1 \in L^\eta(P) \quad \Longleftrightarrow \quad T^1 \in L^{\eta+1}(\hat{P}), \tag{3.2}$$

$$T^1 \in L^2(\hat{P}) \quad \text{when } d_1 \geq 7, \tag{3.3}$$

$$T^1 \in L^4(P) \quad \text{and} \quad T^1 \in L^5(\hat{P}) \quad \text{when } d_1 \geq 13, \tag{3.4}$$

$$\sup_{a\in[0,T^1]} |\chi_a| \in L^2(\hat{P} \times K_0) \quad \text{when } d_1 \geq 7, \tag{3.5}$$

$$\sup_{a\in[0,T^1]} |\chi_a| \in L^4(\hat{P} \times K_0) \quad \text{when } d_1 \geq 13. \tag{3.6}$$

Proof The equivalence (3.2) is an easy consequence of (2.32). With the help of (2.33), we find that $T^1 \in L^1(P)$ when $d_1 \geq 7$ and $T^1 \in L^4(P)$ when $d_1 \geq 13$, and so (3.2) yields (3.3) and (3.4). With the help of the integral representation of $\chi_\cdot$, see (2.13), since $\hat{P} \times K_{0,\omega} \ll P \times K_{0,\omega}$, by (1.3) we see that, for each $\omega \in \Omega$, $\hat{P} \times K_{0,\omega}$-a.s.,

$$\sup_{a\in[0,T^1]} |\chi_a|^2 \leq 2\kappa^2 \big(T^1\big)^2 + 2 \sup_{a\in[0,T^1]} |W_a|^2. \tag{3.7}$$

Taking the $\hat{P} \times K_{0,\omega}$-expectation on both sides of (3.7), we observe that (3.5) follows from (3.3) if we show that, uniformly in ω,

$$E^{\hat{P}\times K_{0,\omega}}\Big[\sup_{a\in[0,T^1]} |W_a|^2\Big] \leq c_5(\varepsilon) < \infty. \tag{3.8}$$

The left-hand side of (3.8) is equal to

$$\sum_{n\geq 1} E^{\hat{P}\times K_{0,\omega}}\Big[\sup_{a\in[0,n]}|W_a|^2, T^1=n\Big]$$
$$\overset{\text{Hölder}}{\leq} \sum_{n\geq 1} E^{\hat{P}\times K_{0,\omega}}\Big[\sup_{a\in[0,n]}|W_a|^{2p}\Big]^{1/p}\hat{P}\big[T^1=n\big]^{1/q} \tag{3.9}$$

with $1<q<\frac{6}{5}$ and p the conjugate exponent. From (2.27) and the definition of $\hat{P}$, see (2.29), we see that

$$\hat{P}[\,\cdot\,]\leq c_1(\varepsilon)^{-1}P[\,\cdot\,]. \tag{3.10}$$

An application of the Burkholder–Davis–Gundy inequality, see p. 166 of [13], yields

$$E^{\hat{P}\times K_{0,\omega}}\Big[\sup_{a\in[0,n]}|W_a|^{2p}\Big]^{1/p}\overset{(3.10)}{\leq} c(\varepsilon,q)E^{P\times K_{0,\omega}}\Big[\sup_{a\in[0,n]}|W_a|^{2p}\Big]^{1/p}\leq c(\varepsilon,q)n,$$

and hence the right-hand side of (3.9) is less or equal to

$$c(\varepsilon,q)\sum_{n\geq 1} n\hat{P}\big[T^1=n\big]^{1/q}=c(\varepsilon,q)\sum_{n\geq 1} n\hat{P}\big[T^1=n\big]^{1/2}\hat{P}\big[T^1=n\big]^{1/q-1/2}. \tag{3.11}$$

From an application of Cauchy–Schwarz' inequality it follows that (3.11) is dominated by

$$c(\varepsilon,q)\bigg\{E^{\hat{P}}\big[(T^1)^2\big]^{1/2}\Big(\sum_{n\geq 1}\hat{P}\big[T^1=n\big]^{2/q-1}\Big)^{1/2}\bigg\}. \tag{3.12}$$

Since $\hat{P}[T^1=n]\leq c_1(\varepsilon)^{-1}P[T^1>n-1]$ holds and $d_1\geq 7$, one easily checks by using (2.33) that the sum in (3.12) with $1<q<\frac{6}{5}$ is bounded by a constant $c(\varepsilon,q)$ and, with (3.3), (3.8) then follows. Equation (3.6) is shown analogously to (3.5) with $1<q<\frac{6}{5}$, using now (3.4) instead of (3.3). □

We now are ready to state our first invariance principle.

Theorem 3.3 *Let us assume $d_1\geq 7$ and (3.1). Under the measure P_0, the $C(\mathbb{R}_+,\mathbb{R}^d)$-valued random variables*

$$B^r_{\cdot}\overset{\text{def}}{=}\frac{1}{\sqrt{r}}X_{r\cdot},\qquad r>0, \tag{3.13}$$

converge in law to a d-dimensional Brownian motion $B_{\cdot}$ with covariance matrix

$$A=E^{\hat{P}}\big[T^1\big]^{-1}\begin{pmatrix}E^{\hat{P}}[T^1]I_{d_1} & 0\\ 0 & E^{\hat{Q}^s}[(Y^s_{T^1})(Y^s_{T^1})^t]\end{pmatrix}\in\mathbb{R}^{d\times d} \tag{3.14}$$

as $r\to\infty$.

Remark 3.4 Before giving the proof of the theorem, let us recall some classical facts about weak convergence on $C(\mathbb{R}_+,\mathbb{R}^d)$ that will be used several times throughout Sect. 3. More details on the following results can be found in Chap. 3 of [9] and in Sect. 3.1 of [29]. Let us consider the space $C(\mathbb{R}_+,\mathbb{R}^d)$ and the metric

$$d(\xi_.,\zeta_.)\stackrel{\text{def}}{=}\sum_{m=1}^{\infty}2^{-m}\sup_{0\le t\le m}\big(|\xi_t-\zeta_t|\wedge 1\big)\le 1,\quad \xi_.,\zeta_.\in C\big(\mathbb{R}_+,\mathbb{R}^d\big).$$

Then $C(\mathbb{R}_+,\mathbb{R}^d)$ with the topology induced by $d(\cdot,\cdot)$ is a Polish space. Suppose that $\xi^n_.$ and $\zeta^n_.$, $n\ge 1$, are two $C(\mathbb{R}_+,\mathbb{R}^d)$-valued sequences on some probability space $(\Xi,\mathcal{D},\mu)$. If $d(\xi^n_.,\zeta^n_.)$ converges in μ-probability to 0, then $(\xi^n_.)_{n\ge1}$ under μ is wce to $(\zeta^n_.)_{n\ge1}$ under μ, see below Remark 3.1 for the meaning of wce. Note that in order to verify the convergence in probability μ of the distance $d(\xi^n_.,\zeta^n_.)$ to 0, it suffices to check that, for any $T>0$ and $\varepsilon>0$,

$$\mu\Big(\sup_{0\le t\le T}\big|\xi^n_t-\zeta^n_t\big|>\varepsilon\Big)\stackrel{n\to\infty}{\longrightarrow}0. \tag{3.15}$$

Proof of Theorem 3.3 Observe that Theorem 3.3 follows if we show that, for integer $n\ge 1$,

$$B^n_.\to B_.\text{ in law under }P_0\text{ as }n\to\infty. \tag{3.16}$$

Indeed, (3.16) implies that, as $s_n\nearrow\infty$, the sequence $[s_n]^{-1/2}X_{[s_n]\cdot}$ and thus $s_n^{-1/2}X_{[s_n]\cdot}$ converges in law to $B_.$, recall (2.1). Therefore, the laws of $s_n^{-1/2}X_{[s_n]\cdot}$ are tight, and hence, by Theorem 2.4.10 of [13], for all $T>0$ and $\varepsilon>0$, there exists an $\eta>0$ such that

$$\sup_{n\ge1}P_0\Bigg[\sup_{\substack{|s-t|\le\eta\\0\le s,t\le T}}\frac{1}{\sqrt{s_n}}|X_{[s_n]t}-X_{[s_n]s}|\ge\varepsilon\Bigg]\le\varepsilon.$$

Since $\sup_{t\le T}|t-\frac{s_n}{[s_n]}t|\stackrel{n\to\infty}{\longrightarrow}0$, we obtain that for large n,

$$P_0\Big[\sup_{0\le t\le T}\frac{1}{\sqrt{s_n}}|X_{[s_n]t}-X_{s_nt}|\ge\varepsilon\Big]\le\varepsilon.$$

In view of Remark 3.4, this shows that $B^{s_n}_.$ converges in law to $B_.$ for any $s_n\nearrow\infty$, which proves Theorem 3.3. For integer $n\ge1$, we introduce the following piece-wise linear processes (recall the definitions (2.46) and (2.48)):

$$\begin{aligned}\bar B^n_.&\stackrel{\text{def}}{=}\frac{1}{\sqrt n}\big\{X_{[n\cdot]}+\big(n\cdot-[n\cdot]\big)(X_{[n\cdot]+1}-X_{[n\cdot]}t)\big\},\\ \bar Z_n(\cdot)&\stackrel{\text{def}}{=}Z_{[n\cdot]}+\big(n\cdot-[n\cdot]\big)(Z_{[n\cdot]+1}-Z_{[n\cdot]}),\\ \bar Z^s_n(\cdot)&\stackrel{\text{def}}{=}Z^s_{[n\cdot]}+\big(n\cdot-[n\cdot]\big)\big(Z^s_{[n\cdot]+1}-Z^s_{[n\cdot]}\big).\end{aligned}\tag{3.17}$$

Note that the processes $\bar{B}^n_\cdot$, $\frac{1}{\sqrt{n}}\bar{Z}_n(\cdot)$ and $\frac{1}{\sqrt{n}}\bar{Z}^s_n(\cdot)$ are the polygonal interpolations of $(X_k)_{k\geq 0}$, $(Z_k)_{k\geq 0}$, and $(Z^s_k)_{k\geq 0}$, respectively, which are then rescaled in time and space as in the definition of $B^n_\cdot$ for integers $n \geq 1$, see (3.13).

Lemma 3.5 *$(B^n_\cdot)_{n\geq 1}$ under P_0 is wce to $(\frac{1}{\sqrt{n}}\bar{Z}^s_n(\cdot))_{n\geq 1}$ under Q^s.*

Proof As a first step, we show that

$$\big(B^n_\cdot\big)_{n\geq 1} \text{ under } P_0 \text{ is wce to } \big(\bar{B}^n_\cdot\big)_{n\geq 1} \text{ under } P_0. \tag{3.18}$$

In view of Remark 3.4, see in particular (3.15), it suffices to prove that for any $T > 0$, the sequence of random variables $\sup_{0\leq t\leq T} |B^n_t - \bar{B}^n_t|$ converges in P_0-probability to 0 as $n \to \infty$. Indeed, the process $(W'_t)_{t\geq 0}$ defined below (2.68) is a d-dimensional Brownian motion under $P_{0,\omega}$, and so, for $T > 0$ and $\varepsilon > 0$, when n is large uniformly in ω,

$$\begin{aligned}
&P_{0,\omega}\Big[\sup_{0\leq t\leq T}\big|B^n_t - \bar{B}^n_t\big| \geq 4\varepsilon\Big] \\
&\quad\leq P_{0,\omega}\Big[\sup_{\substack{k=0,\dots,[Tn]\\ 0\leq a\leq 1}} |X_{k+a} - X_k| \geq 2\varepsilon\sqrt{n}\Big] \\
&\quad= P_{0,\omega}\Bigg[\sup_{\substack{k=0,\dots,[Tn]\\ 0\leq a\leq 1}} \bigg|\int_k^{k+a} b(X_s,\omega)\,ds + W'_{k+a} - W'_k\bigg| \geq 2\varepsilon\sqrt{n}\Bigg] \\
&\quad\leq c(1+Tn)\exp\bigg\{-\frac{\varepsilon^2}{2d^2}n\bigg\},
\end{aligned}$$

where we used (1.3) and Bernstein's inequality in the last line, and (3.18) follows. From the identities in law stated in (1) of Theorem 2.2 and Proposition 2.8 we immediately deduce that

$$\big(\bar{B}^n_\cdot\big)_{n\geq 1} \text{ under } P_0 \text{ is identical in law to } \bigg(\frac{1}{\sqrt{n}}\bar{Z}_n(\cdot)\bigg)_{n\geq 1} \text{ under } Q^0. \tag{3.19}$$

A combination of (3.18) and (3.19) yields Lemma 3.5 once we show that

$$\bigg(\frac{1}{\sqrt{n}}\bar{Z}_n(\cdot)\bigg)_{n\geq 1} \text{ under } Q^0 \text{ is wce to } \bigg(\frac{1}{\sqrt{n}}\bar{Z}^s_n(\cdot)\bigg)_{n\geq 1} \text{ under } Q^s. \tag{3.20}$$

As in the proof of Theorem 2.11, we define the processes $\bar{Z}_n(\cdot)$ and $\bar{Z}^s_n(\cdot)$ on a common probability space, see (2.70) and below. Then we can again use the strategy discussed in Remark 3.4 to prove (3.20). In the notation (2.70)–(2.74), using the fact that (2.75) holds true, we find for $T > 0$ that Q-a.s.,

$$\sup_{0\leq t\leq T}\frac{1}{\sqrt{n}}\big|\bar{Z}_n(t)\circ\pi^0 - \bar{Z}^s_n(t)\circ\pi^s\big| \leq \sup_{0\leq t\leq \frac{T^1}{n}}\frac{1}{\sqrt{n}}\big|\bar{Z}_n(t)\circ\pi^0 - \bar{Z}^s_n(t)\circ\pi^s\big|. \tag{3.21}$$

Since $\bar{Z}_n(t)\circ\pi^0 - \bar{Z}^s_n(t)\circ\pi^s$, $t\in[0,\frac{T^1}{n}]$, is a continuous process which is piecewise linear between the times $0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{T^1}{n}$, we find that the right-hand side of (3.21) is equal to

$$\sup_{k=0,\frac{1}{n},\dots,\frac{T^1}{n}} \frac{1}{\sqrt{n}}\big|\bar{Z}_n(k)\circ\pi^0 - \bar{Z}^s_n(k)\circ\pi^s\big| = \sup_{k=0,\dots,T^1}\frac{1}{\sqrt{n}}\big|Z(k)\circ\pi^0 - Z^s(k)\circ\pi^s\big|,$$

which converges Q-a.s. to zero as $n\to\infty$, since $Q[T^1<\infty]=1$, see (2.26) and (2.28). This concludes the proof of (3.20) and thus of Lemma 3.5. □

Let us define an integer-valued function $0\le\varphi(t)$ tending to infinity P-a.s. such that

$$T^{\varphi(t)} \le t < T^{\varphi(t)+1} \quad \text{for all } t\ge 0 \tag{3.22}$$

and

$$\Sigma_m \stackrel{\text{def}}{=} Z^s_{T^m} - Z^s_{T^0}, \quad m\ge 0. \tag{3.23}$$

Furthermore, let us introduce the polygonal interpolation of $\Sigma_m, m\ge 0$:

$$\bar{\Sigma}_. \stackrel{\text{def}}{=} \Sigma_{[\cdot]} + \big(\cdot - [\cdot]\big)(\Sigma_{[\cdot]+1} - \Sigma_{[\cdot]}), \tag{3.24}$$

and for integer $n\ge 1$,

$$\bar{\Sigma}^{\varphi}_n(\cdot) \stackrel{\text{def}}{=} \Sigma_{\varphi(n\cdot)} + \big(n\cdot - [n\cdot]\big)(\Sigma_{\varphi(n\cdot+1)} - \Sigma_{\varphi(n\cdot)}), \tag{3.25}$$

which is constant and equal to $\Sigma_{\varphi(T^k)} = \Sigma_k$ on the time interval $[\frac{T^k}{n}, \frac{T^{k+1}}{n} - \frac{1}{n})$, $k\ge 0$, and linear on the interval $[\frac{T^{k+1}}{n} - \frac{1}{n}, \frac{T^{k+1}}{n})$, interpolating the points Σ_k and Σ_{k+1}.

Lemma 3.6 *$(\frac{1}{\sqrt{n}}\bar{Z}^s_n(\cdot))_{n\ge 1}$ under Q^s is wce to $(\frac{1}{\sqrt{n}}\bar{\Sigma}^{\varphi}_n(\cdot))_{n\ge 1}$ under Q^s.*

Proof In view of Remark 3.4, it suffices to show that, for any $T>0$ and $\varepsilon>0$, the following probability converges to 0 as $n\to\infty$:

$$\begin{aligned} &Q^s\bigg[\sup_{0\le t\le T}\frac{1}{\sqrt{n}}\big|\bar{Z}^s_n(t) - \bar{\Sigma}^{\varphi}_n(t)\big| > 4\varepsilon\bigg] \\ &\quad \le \underbrace{Q^s\bigg[\sup_{\substack{k=0,\dots,[Tn]+1\\ a\in[0,T^{k+1}-T^k]}}\big|Z^s_{T^k+a} - Z^s_{T^k}\big| > \varepsilon\sqrt{n}\bigg]}_{\stackrel{\text{def}}{=} A_n}. \end{aligned} \tag{3.26}$$

Since the event A_n is invariant under the shift Θ_{T^0} and the image of Q^s under Θ_{T^0} is $E^{\hat{Q}^s}[\cdot, T^1]/E^{\hat{P}}[T^1]$, see (2.60), it follows by Cauchy–Schwarz' inequality that

$$Q^s[A_n] = Q^s\big[\Theta^{-1}_{T^0}(A_n)\big] \le E^{\hat{P}}\big[(T^1)^2\big]^{1/2}\hat{Q}^s[A_n]^{1/2}/E^{\hat{P}}\big[T^1\big], \tag{3.27}$$

where $E^{\hat{P}}[(T^1)^2] < \infty$, see (3.3). Thus, Lemma 3.6 will follow once we show that

$$\lim_{n\to\infty} \hat{Q}^s[A_n] = 0. \tag{3.28}$$

Using (2.56) and the fact that $\hat{\Theta}_k$ preserves $\hat{Q}^s$, see the proof of Proposition 2.10, we find that

$$\begin{aligned}\hat{Q}^s[A_n] &\le (2+Tn)\hat{Q}^s\Big[\sup_{a\in[0,T^1]}\big|Z_a^s\big| > \varepsilon\sqrt{n}\Big]\\ &\le \frac{2+Tn}{\varepsilon^2 n}E^{\hat{Q}^s}\Big[\sup_{a\in[0,T^1]}\big|Z_a^s\big|^2,\ \sup_{a\in[0,T^1]}\big|Z_a^s\big| > \varepsilon\sqrt{n}\Big].\end{aligned} \tag{3.29}$$

From (3.5) and Remark 2.7 it follows that the last expression vanishes as $n\to\infty$, and hence (3.28) holds true. This finishes the proof of Lemma 3.6. □

Lemma 3.7 *Under Q^s, $(\frac{1}{\sqrt{n}}\bar{\Sigma}_{n\cdot})_{n\ge 1}$ converges in law to $\sqrt{E^{\hat{P}}[T^1]}B_.$ as $n\to\infty$.*

Before proving Lemma 3.7, let us explain how we conclude the proof of Theorem 3.3. Once we show that

$$\Big(\frac{1}{\sqrt{n}}\bar{\Sigma}_n^{\varphi}(\cdot)\Big)_{n\ge 1} \text{ under } Q^s \text{ is wce to } \Big(\frac{1}{\sqrt{n}}\bar{\Sigma}_{n\cdot/E^{\hat{P}}[T^1]}\Big)_{n\ge 1} \text{ under } Q^s, \tag{3.30}$$

we find with Lemma 3.7 and a transformation of time that the first sequence in (3.30) converges weakly to $B_.$ as $n\to\infty$ and hence by Lemmas 3.6 and 3.5 we deduce that (3.16) holds, which finishes the proof of Theorem 3.3.

For the proof of (3.30), first note that since $T^n - T^0$ is invariant under Θ_{T^0}, we find by similar arguments to those leading to (3.27) that the convergence in (2.78) holds true Q^s-a.s. and not only $\hat{P}$-a.s. It follows that $\frac{\varphi(t)}{t}\to E^{\hat{P}}[T^1]^{-1}$ Q^s-a.s. and hence by Lemma 9.2 on p. 572 of [11]:

$$\text{for all } T\ge 0,\ Q^s\text{-a.s.},\qquad \sup_{0\le t\le T}\left|\frac{\varphi(nt)}{n} - \frac{t}{E^{\hat{P}}[T^1]}\right| \overset{n\to\infty}{\longrightarrow} 0, \tag{3.31}$$

and so, for $\varepsilon>0$, $\eta>0$, $T>0$, and n large enough,

$$Q^s\left[\sup_{0\le t\le T}\left|\frac{\varphi(nt)}{n} - \frac{t}{E^{\hat{P}}[T^1]}\right| \ge \eta\right] \le \varepsilon. \tag{3.32}$$

Furthermore, from Lemma 3.7 we infer that the laws of $n^{-1/2}\bar{\Sigma}_{n\cdot}$ under Q^s are tight and hence, for all $T>0$ and $\varepsilon>0$, there exists $\eta>0$ such that

$$\sup_{n\ge 1} Q^s\Big[\sup_{\substack{|s-t|\le\eta\\ 0\le s,t\le T}} \frac{1}{\sqrt{n}}\big|\bar{\Sigma}_{nt} - \bar{\Sigma}_{ns}\big| \ge \varepsilon\Big] \le \varepsilon, \tag{3.33}$$

see Theorem 2.4.10 of [13]. Together with (3.32), we thus obtain that, for arbitrary $\varepsilon > 0$ and $T > 0$,

$$Q^s\Big[\sup_{0\le t\le T}\frac{1}{\sqrt{n}}\big|\bar{\Sigma}_{nt/E^{\hat{P}}[T^1]}-\bar{\Sigma}_{\varphi(nt)}\big|\ge\varepsilon\Big]\le 2\varepsilon \tag{3.34}$$

for sufficiently large n. In order to prove (3.30) it suffices to show that, for $T > 0$ and $\varepsilon > 0$, the following probability tends to zero with n, see Remark 3.4:

$$\begin{aligned}
&Q^s\Big[\sup_{0\le t\le T}\big|\bar{\Sigma}_{nt/E^{\hat{P}}[T^1]}-\bar{\Sigma}^{\varphi}_n(t)\big|>2\varepsilon\sqrt{n}\Big]\\
&\quad\le Q^s\Big[\sup_{0\le t\le T}|\bar{\Sigma}_{nt/E^{\hat{P}}[T^1]}-\bar{\Sigma}_{\varphi(nt)}|>\varepsilon\sqrt{n}\Big]\\
&\qquad+Q^s\Big[\sup_{0\le t\le T}\big|\bar{\Sigma}_{\varphi(nt)}-\bar{\Sigma}^{\varphi}_n(t)\big|>\varepsilon\sqrt{n}\Big].
\end{aligned}$$

The first expression on the right-hand side of the above inequality vanishes as $n\to\infty$, due to (3.34). Moreover, it can easily be seen that the second expression on the right-hand side of the above inequality is less or equal to $Q^s[A_n]$, see (3.26) for the definition of A_n, which tends to 0 as $n\to\infty$, see (3.27) and (3.28). This finishes the proof of (3.30) and hence of Theorem 3.3. □

Proof of Lemma 3.7 Let us denote by $\bar{\Sigma}^{d_1}_{\cdot}$ and $\bar{\Sigma}^{d_2}_{\cdot}$ the first d_1 respectively the last d_2 components of the process $\bar{\Sigma}_{\cdot}$ defined in (3.24). Note that Lemma 3.7 follows from the following two statements:

$$\begin{aligned}&\text{Under } P,\ \text{the sequence } \tfrac{1}{\sqrt{n}}\bar{\Sigma}^{d_1}_{n\cdot},\ n\ge 1,\ \text{converges in law to a } d_1\text{-dimensional}\\ &\text{Brownian motion with covariance matrix } E^{\hat{P}}[T^1]I_{d_1}\ \text{as } n\to\infty,\end{aligned}\tag{3.35}$$

and for P-a.e. $(w,\lambda_\cdot)\in W$,

$$\begin{aligned}&\text{Under the measure } M^s,\ \text{the sequence } \tfrac{1}{\sqrt{n}}\bar{\Sigma}^{d_2}_{n\cdot},\ n\ge 1,\ \text{converges in}\\ &\text{law to a } d_2\text{-dimensional Brownian motion with covariance matrix}\\ &E^{\hat{Q}^s}[(Y^s_{T^1})(Y^s_{T^1})^t]\in\mathbb{R}^{d_2\times d_2}\ (\text{independent of } (w,\lambda_\cdot))\ \text{as } n\to\infty.\end{aligned}\tag{3.36}$$

Indeed, from (3.35) and (3.36) we can easily deduce that under Q^s, the laws of $n^{-1/2}\bar{\Sigma}_{n\cdot}$, $n\ge 1$, are tight, see Theorems 2.4.7 and 2.4.10 of [13]. Therefore, in order to prove Lemma 3.7, it suffices to show the weak convergence of all finite-dimensional distributions of $n^{-1/2}\bar{\Sigma}_{n\cdot}$ to the finite-dimensional distributions of $\sqrt{E^{\hat{P}}[T^1]}B_\cdot$ as $n\to\infty$, see Theorem 2.4.15 in [13]. But this can easily be inferred from (3.35) and (3.36) with the help of characteristic functions.

Now, let us explain how to see (3.35). Similarly as in the proof of (3.30), we first note that, for $\varepsilon>0$, $\eta>0$, $T>0$, and n large enough,

$$P\Big[\sup_{0\le t\le T}\Big|\frac{T^{[nt]}}{E^{\hat{P}}[T^1]n}-t\Big|\ge\eta\Big]\le\varepsilon. \tag{3.37}$$

Furthermore, by the definition and self-similarity of Brownian motion we know that under P, the processes $n^{-1/2}X^1_{E^{\hat{P}}[T^1]n\cdot}, n \geq 1$, are distributed as a d_1-dimensional Brownian motion with covariance matrix $E^{\hat{P}}[T^1]I_{d_1}$, and hence their laws are tight. So, we can derive the same estimate as in (3.33) but for the process $X^1_{E^{\hat{P}}[T^1]n\cdot}$. Together with (3.37), we thus obtain that

$$P\Big[\sup_{0\leq t\leq T}\big|X^1_{T^{[nt]}} - X^1_{E^{\hat{P}}[T^1]nt}\big| \geq \varepsilon\sqrt{n}\Big] \leq 2\varepsilon \tag{3.38}$$

for sufficiently large n. Pick $T>0$ and $\varepsilon>0$, and then observe that

$$\begin{aligned} &P\Big[\sup_{0\leq t\leq T}\big|\bar{\Sigma}^{d_1}_{nt} - X^1_{E^{\hat{P}}[T^1]nt}\big| > 3\varepsilon\sqrt{n}\Big] \\ &\quad\leq P\Big[\sup_{0\leq t\leq T}\big|X^1_{T^{[nt]}} - X^1_{E^{\hat{P}}[T^1]nt}\big| > \varepsilon\sqrt{n}\Big] \\ &\qquad + P\Big[\sup_{\substack{k=0,\dots,[Tn]\\ a\in[0,T^{k+1}-T^k]}}\big|X^1_{T^k+a} - X^1_{T^k}\big| > \varepsilon\sqrt{n}\Big]. \end{aligned}$$

The first term after the above inequality tends to zero with n due to (3.38), whereas the second term is dominated by $Q^s(A_n)$ and thus converges to zero as well as $n\to\infty$. In view of Remark 3.4, this proves (3.35).

We now prove (3.36). Note that for $\hat{P}$-a.e. $(w,\lambda_\cdot)\in W\cap\{0\in\mathcal{C}\}$, under M^s, the increments $Y^s_{T^n} - Y^s_{T^{n-1}}$, $n\geq 1$, are independent, see (2.45) and (2.49), with mean zero, which is a consequence of the symmetry assumption (3.1). Indeed, for $\hat{P}$-a.e. $(w,\lambda_\cdot)\in W\cap\{0\in\mathcal{C}\}$,

$$E^{M^s}\big[Y^s_{T^n} - Y^s_{T^{n-1}}\big] = \big(E^{K_0}\big[X^2_{T^1}\big]\big)\circ\hat{\theta}_{n-1} \overset{(2.17)}{=} \mathbb{E}\big[E^{K_{0,\omega}}\big[X^2_{T^1}\big]\big]\circ\hat{\theta}_{n-1}. \tag{3.39}$$

In Remark 2.4 we have seen that we can write

$$E^{K_{0,\omega}}\big[X^2_{T^1}\big] = \int_{\mathbb{R}^{d_2}}\cdots\int_{\mathbb{R}^{d_2}} dy_1\cdots dy_{T^1}\prod_{k=0}^{T^1-1} h\big(w(k+\cdot)-w(k),\lambda_k,y_k,y_{k+1},\hat{\omega}_k\big)y_{T^1} \tag{3.40}$$

with $\hat{\omega}_k = \tau_{(w(k),0)}(\omega)$, $k=0,\dots,T^1-1$, and $y_0 := 0$. Let us denote with $h^{(\mathcal{R})}$ the analogue to h, see (2.19), defined via the transition density $p^{(\mathcal{R})}_{w,\omega}(1,\cdot,\cdot)$ for $\omega\in\Omega$, $w\in W^{d_1}_0$, attached by (2.5) and (2.6) to the drift $-b^*((w(\cdot),-\cdot),\omega)$, see also (1.1). Since $(X^2_t)_{t\in[0,1]}$ under $\tilde{P}_{y_k,y_{k+1}}$ has the same law as $(-X^2_t)_{t\in[0,1]}$ under $\tilde{P}_{-y_k,-y_{k+1}}$, see below (2.3) for the definition of $\tilde{P}_{\cdot,\cdot}$, we see that, for $k=0,\dots,T^1-1$,

$$p^{(\mathcal{R})}_{w,\omega}(1,-y_k,-y_{k+1}) = p_{w,\omega}(1,y_k,y_{k+1})$$

and hence

$$h^{(\mathcal{R})}\big(w(k+\cdot)-w(k),\lambda_k,-y_k,-y_{k+1},\hat{\omega}_k\big) = h\big(w(k+\cdot)-w(k),\lambda_k,y_k,y_{k+1},\hat{\omega}_k\big). \tag{3.41}$$

With the help of Theorem 44 on p. 158 in [22] one can see that for fixed $(w, \lambda_\cdot) \in W$ and $y_k, y_{k+1} \in \mathbb{R}^{d_2}$, the expression on the left-hand side of (3.41) is a measurable function of $\mathcal{R}(b(\mathcal{R}(\cdot), \omega))$, see (1.1). So (3.41), together with our symmetry assumption (3.1), implies that under $\mathbb{P}$, the product in (3.40) is identical in law to

$$\prod_{k=0}^{T^1-1} h\big(w(k+\cdot)-w(k), \lambda_k, -y_k, -y_{k+1}, \hat{\omega}_k\big)$$

and so a transformation of variables $(y_1, \dots, y_{T^1}) \mapsto (-y_1, \dots, -y_{T^1})$ and Fubini's Theorem then yield $E^{K_0}[X^2_{T^1}] = -E^{K_0}[X^2_{T^1}] = 0$. Hence, in view of (3.39), for $\hat{P}$-a.e. $(w, \lambda_\cdot) \in W \cap \{0 \in \mathcal{C}\}$,

$$E^{M^s}\big[Y^s_{T^n} - Y^s_{T^{n-1}}\big] = 0,$$

since $\hat{\theta}_{n-1}$ preserves $\hat{P}$. Furthermore,

$$\begin{aligned} E^{\hat{P}}\big[E^{M^s}\big[\big|Y^s_{T^n} - Y^s_{T^{n-1}}\big|^2\big]\big] &\overset{(2.43),(2.48)}{\leq} E^{\hat{Q}^s}\big[\big|Z^s_{T^n} - Z^s_{T^{n-1}}\big|^2\big] \\ &\overset{(2.56)}{=} E^{\hat{Q}^s}\big[\big|Z^s_{T^1}\big|^2 \circ \hat{\Theta}_{n-1}\big] < \infty, \end{aligned} \tag{3.42}$$

since $\hat{\Theta}_{n-1}$ preserves $\hat{Q}^s$ and because of the integrability property (3.5) and Remark 2.7. In particular, it follows that, for $\hat{P}$-a.e. $(w, \lambda_\cdot) \in W \cap \{0 \in \mathcal{C}\}$,

$$Y^s_{T^n} - Y^s_{T^{n-1}} \in L^2\big(M^s(w, \lambda_\cdot)\big). \tag{3.43}$$

Note that $(W \cap \{0 \in \mathcal{C}\}, \hat{\theta}_1, \hat{P})$ is ergodic as a consequence of the ergodicity of (W, θ_1, P), see (34) on p. 357 in [20]. An application of an invariance principle for vector-valued, square-integrable martingale differences, see Theorem 5.1, shows that, for $\hat{P}$-a.e. $(w, \lambda_\cdot) \in W \cap \{0 \in \mathcal{C}\}$, under the measure $M^s(w, \lambda_\cdot)$, the $C(\mathbb{R}_+, \mathbb{R}^{d_2})$-valued random variables $n^{-1/2}\bar{\Sigma}^{d_2}_{n\cdot}$, $n \geq 1$, converge weakly to a d_2-dimensional Brownian motion with covariance matrix as in (3.36) as $n \to \infty$. Note that in fact under the measure $M^s(w, \lambda_\cdot)$ the increments $Y_{T^n} - Y_{T^{n-1}}, n \geq 1$, are independent and hence the standard functional central limit theorem, which is an immediate consequence of Theorem 5.1, can be applied. The ergodicity of $(W \cap \{0 \in \mathcal{C}\}, \hat{\theta}_1, \hat{P})$ and the integrability property (3.42) are used to show that conditions (5.1) and (5.2) are satisfied. Since for a continuous, bounded function f on $W^{d_2}_+$, the random variable $E^{M^s}[f(n^{-1/2}\bar{\Sigma}^{d_2}_{n\cdot}))]$ is invariant under θ_{T^0} and since the image of P under θ_{T^0} is absolutely continuous with respect to $\hat{P}$, see (2.32), it follows that (3.36) holds in fact for P-a.e. $(w, \lambda_\cdot) \in W$. This finishes the proof of Lemma 3.7. □

The next theorem shows us that our model also contains examples of diffusions in random environment with possibly ballistic behavior when $d_1 \geq 13$, satisfying an invariance principle, recall (1.1)–(1.7).

Theorem 3.8 *Let $d_1 \geq 13$, and recall the definition of v in (2.66). Under the measure P_0, the $C(\mathbb{R}_+, \mathbb{R}^d)$-valued random variables*

$$B^r_\cdot \stackrel{\text{def}}{=} \frac{1}{\sqrt{r}}(X_{r\cdot} - vr\cdot), \quad r > 0,$$

converge in law to a d-dimensional Brownian motion $B_\cdot$ with covariance matrix A given in (3.52) as $r \to \infty$.

Proof As in the proof of Theorem 3.3, we can show that it suffices to prove that, for integer $n \geq 1$,

$$B^n_\cdot \to B_\cdot \text{ in law under } P_0 \text{ as } n \to \infty, \tag{3.44}$$

see (3.16). By similar arguments as in the proof of Lemma 3.5 we find that

$$\left(\bar{B}^n_\cdot - v\sqrt{n}\,\cdot\right)_{n\geq 1} \text{ under } P_0 \text{ is wce to } \left(\frac{1}{\sqrt{n}}\left(\bar{Z}^s_n(\cdot) - vn\cdot\right)\right)_{n\geq 1} \text{ under } Q^s, \tag{3.45}$$

see (3.19) and (3.20). Together with (3.18), we see that (3.44) follows once we show that

$$\frac{1}{\sqrt{n}}\left(\bar{Z}^s_n(\cdot) - vn\cdot\right) \to B_\cdot \text{ in law under } Q^s \quad \text{as } n \to \infty, \tag{3.46}$$

see (3.17) for the definition of $\bar{Z}^s_n(\cdot)$ and $\bar{B}^n_\cdot$. In the notation

$$\mathcal{Z} \stackrel{\text{def}}{=} Z^s_1 - E^{Q^s}\left[Z^s_1\right] = Z^s_1 - v, \quad \text{we have that} \quad Z^s_n - vn \stackrel{(2.57)}{=} \sum_{k=0}^{n-1} \mathcal{Z} \circ \Theta_k, \tag{3.47}$$

recall (2.66). We know from (2.60) that

$$\mathcal{Z} \in L^m\left(Q^s\right) \quad \text{for all } m \in [1, \infty). \tag{3.48}$$

For integers $k \geq 0$, on the space Γ^s, we introduce the filtration

$$\mathcal{G}_k \stackrel{\text{def}}{=} \sigma\left(Z^s_{n+1} - Z^s_n \text{ for all } n \in \mathbb{Z} \text{ with } n < k\right). \tag{3.49}$$

Identity (2.56) implies that, for $k \geq 0$,

$$f \text{ is } \mathcal{G}_0\text{-measurable} \quad \Longleftrightarrow \quad f \circ \Theta_k \text{ is } \mathcal{G}_k\text{-measurable}, \tag{3.50}$$

and thus by stationarity, see (2.59), we have that, for $g \in L^1(Q^s)$,

$$Q^s\text{-a.s.} \quad E^{Q^s}[g \circ \Theta_k \mid \mathcal{G}_k] = E^{Q^s}[g \mid \mathcal{G}_0] \circ \Theta_k. \tag{3.51}$$

The following adaptation of Gordin's method will play the key role in the proof.

Lemma 3.9 *There is a $G \in L^2(\Gamma^s, \mathcal{G}_0, Q^s)$ such that*

$$M_n \stackrel{\text{def}}{=} G \circ \Theta_n - G + Z_n^s - vn = \sum_{k=0}^{n-1} (G \circ \Theta_1 - G + \mathcal{Z}) \circ \Theta_k, \quad n \geq 0,$$

is a $(\mathcal{G}_n)$-martingale.

Before we prove Lemma 3.9, let us explain how we conclude the proof of Theorem 3.8 from it. Using the stationarity of Θ_1 under Q^s, see (2.59), and applying Chebychev's inequality we obtain that, for $T > 0$ and $\varepsilon > 0$,

$$Q^s\Big[\sup_{k=0,\dots,[Tn]+1} |G \circ \Theta_k| > \varepsilon\sqrt{n}\Big] \leq \frac{Tn+2}{\varepsilon^2 n} E^{Q^s}\big[|G|^2, |G| > \varepsilon\sqrt{n}\big] \stackrel{n\to\infty}{\longrightarrow} 0,$$

so that, in view of Remark 3.4, one easily finds that $n^{-1/2}(\bar{Z}_n^s(\cdot) - vn\cdot)_{n\geq 1}$ under Q^s is wce to the rescaled polygonal interpolation of the process $M_k, k \geq 1$, defined analogously to $\bar{B}^n_{\cdot}$ in (3.17), under Q^s. Since M_n is a martingale with ergodic, square-integrable increments, it follows from Theorem 5.2, see Appendix, that under the measure Q^s, the rescaled polygonal interpolation of $M_k, k \geq 1$, converges in law to a d-dimensional Brownian motion with covariance matrix

$$A = E^{Q^s}\big[(G \circ \Theta_1 - G + \mathcal{Z})(G \circ \Theta_1 - G + \mathcal{Z})^t\big] \tag{3.52}$$

as $n \to \infty$. This concludes the proof of (3.46). □

Proof of Lemma 3.9 First we explain how our claim follows once we show that

$$\sum_{k\geq 0} \big\| E^{Q^s}\big[(H \mathbb{1}_{\{0\in\mathcal{C}\}}) \circ \Theta_k \mid \mathcal{G}_0\big]\big\|_2 < \infty, \tag{3.53}$$

where in the previous notation, see (3.47),

$$H \stackrel{\text{def}}{=} \sum_{k=0}^{T^1-1} \mathcal{Z} \circ \Theta_k = \sum_{k=0}^{T^1-1} Z_1^s \circ \Theta_k - vT^1 \stackrel{(2.57)}{=} Z_{T^1}^s - vT^1. \tag{3.54}$$

Note that $H \in L^2(Q^s)$. Indeed,

$$\begin{aligned} E^{Q^s}\big[|H|^2\big] &\leq E^{Q^s}\Bigg[(T^1)^2 \sum_{k=0}^{T^1-1} |\mathcal{Z} \circ \Theta_k|^2\Bigg] = \sum_{n\geq 1} n^2 \sum_{k=0}^{n-1} E^{Q^s}\big[|\mathcal{Z} \circ \Theta_k|^2, T^1 = n\big] \\ &\stackrel{\text{Hölder}}{\leq} \sum_{n\geq 1} n^2 \sum_{k=0}^{n-1} E^{Q^s}\big[|\mathcal{Z} \circ \Theta_k|^{2p}\big]^{1/p} P\big[T^1 = n\big]^{1/q} \end{aligned}$$

with $1 < q < 9/8$ and p the conjugate exponent. Since $P[T^1 = n] \leq P[T^1 > n-1]$ and $E^{Q^s}[|\mathcal{Z} \circ \Theta_k|^{2p}] \leq c(p) < \infty$ by (2.59) and (3.48), we conclude with the help of (2.33) that the right-hand side of the above inequality is finite when $d_1 \geq 13$.

For $m \geq 1$, we define

$$G^m \stackrel{\text{def}}{=} E^{Q^s}[H \mid \mathcal{G}_0] + \sum_{k=1}^{m-1} E^{Q^s}\big[(H\mathbb{1}_{\{0\in\mathcal{C}\}}) \circ \Theta_k \mid \mathcal{G}_0\big]. \tag{3.55}$$

Then G^m converges in $L^2(Q^s)$ to $G \in L^2(\Gamma^s, \mathcal{G}_0, Q^s)$ because of (3.53). Moreover, for $m \geq 1$, we define $N_m = N((w, \lambda_.); [1, m-1]) + 1$ in the notation of (2.25), so that

$$\sum_{k=0}^{T^{N_m}-1} \mathcal{Z} \circ \Theta_k = H + \sum_{k=1}^{m-1} (H\mathbb{1}_{\{0\in\mathcal{C}\}}) \circ \Theta_k. \tag{3.56}$$

By stationarity, see (2.59), we find that, for $n \geq 0$,

$$G \circ \Theta_n = \lim_{m\to\infty} G^m \circ \Theta_n \stackrel{(3.51),(3.56)}{=} \lim_{m\to\infty} E^{Q^s}\left[\left(\sum_{k=0}^{T^{N_m}-1} \mathcal{Z} \circ \Theta_k\right) \circ \Theta_n \,\middle|\, \mathcal{G}_n\right], \tag{3.57}$$

where the above limits are in $L^2(\Gamma^s, \mathcal{G}_n, Q^s)$. This yields for $n \geq 1$,

$$\begin{aligned} &E^{Q^s}[M_{n+1} - M_n \mid \mathcal{G}_n] \\ &\quad= \lim_{m\to\infty} E^{Q^s}\Bigg[\left(\sum_{k=0}^{T^{N_m}-1} \mathcal{Z} \circ \Theta_k\right) \circ \Theta_{n+1} + \mathcal{Z} \circ \Theta_n \\ &\qquad - \left(\sum_{k=0}^{T^{N_m}-1} \mathcal{Z} \circ \Theta_k\right) \circ \Theta_n \,\Bigg|\, \mathcal{G}_n\Bigg], \end{aligned}$$

where the limit is in $L^2(\Gamma^s, \mathcal{G}_n, Q^s)$. With the observation that

$$T^{N_m} \circ \theta_1 = \begin{cases} T^{N_m} - 1 & \text{on } \{m \notin \mathcal{C}\}, \\ T^{N_m+1} - 1 & \text{on } \{m \in \mathcal{C}\}, \end{cases}$$

we find that the quantity under the conditional expectation is equal to

$$\left(\sum_{k=0}^{T^{N_m}\circ\theta_1} \mathcal{Z} \circ \Theta_k - \sum_{k=0}^{T^{N_m}-1} \mathcal{Z} \circ \Theta_k\right) \circ \Theta_n = (\mathbb{1}_{\{m\in\mathcal{C}\}} H \circ \Theta_m) \circ \Theta_n = (H\mathbb{1}_{\{0\in\mathcal{C}\}}) \circ \Theta_{n+m}.$$

As an L^2-limit,

$$\begin{aligned} &\lim_{m\to\infty} E^{Q^s}\big[(H\mathbb{1}_{\{0\in\mathcal{C}\}}) \circ \Theta_{n+m} \mid \mathcal{G}_n\big] \\ &\quad\stackrel{(3.51)}{=} \lim_{m\to\infty} E^{Q^s}\big[(H\mathbb{1}_{\{0\in\mathcal{C}\}}) \circ \Theta_m \mid \mathcal{G}_0\big] \circ \Theta_n \stackrel{(3.53)}{=} 0, \end{aligned}$$

thus proving that M_n is a $(\mathcal{G}_n)$-martingale.

It now remains to prove (3.53). We consider $B \in L^2(\Gamma^s, \mathcal{G}_0, Q^s)$ with L^2-norm $\|B\|_2 = 1$. Note that B can be considered as a function of $(w, \lambda_\cdot)$ and $(u_m, \omega_m)_{m\le 0}$. Then it follows that, for fixed $(w, \lambda_\cdot) \in W$, see (2.26), the random vectors B and

$$\sum_{k=T^m}^{T^{m+1}-1} \mathcal{Z} \circ \Theta_k = \left(X^1_{T^{m+1}} - X^1_{T^m}, u_m\left(T^{m+1} - T^m\right)\right) - v\left(T^{m+1} - T^m\right) \quad \text{for } m \ge 1$$

are independent under the measure M^s, see (2.45). With these considerations in mind, we find that, for integer $p \ge 1$,

$$\begin{aligned} E^{Q^s}\left[(H\mathbb{1}_{\{0\in\mathcal{C}\}}) \circ \Theta_p \cdot B\right] &= \sum_{m\ge 1} E^{Q^s}\left[\left(\sum_{k=T^m}^{T^{m+1}-1} \mathcal{Z} \circ \Theta_k\right) \cdot B, T^m = p\right] \\ &= \sum_{m\ge 1} E^P\Bigg[E^{M^s}\left[\sum_{k=T^m}^{T^{m+1}-1} \mathcal{Z} \circ \Theta_k\right] \\ &\qquad \times E^{M^s}[B], T^m = p\Bigg] \\ &= E^P\left[\left(E^{M^s}[H]\mathbb{1}_{\{0\in\mathcal{C}\}}\right) \circ \theta_p \, E^{M^s}[B]\right]. \end{aligned} \tag{3.58}$$

Then observe that we can find measurable functions φ and ψ such that

$$\begin{aligned} E^{M^s}[H]\mathbb{1}_{\{0\in\mathcal{C}\}} &= \varphi\left(T^1, \left(X^1_t\right)_{t\ge 0}, (\Lambda_n)_{n\ge 0}\right)\mathbb{1}_{\{0\in\mathcal{C}\}}, \\ E^{M^s}[B] &= \psi\left(T^0, \left(X^1_t\right)_{t\le 0}, (\Lambda_n)_{n\le -1}\right), \end{aligned} \tag{3.59}$$

recall the definition of Λ_n above (2.10). The reason why $E^{M^s}[B]$ depends only on T^0, $(X^1_t)_{t\le 0}$, and $(\Lambda_n)_{n\le -1}$, whereas the involved cut times $T^k, k \le -1$, are based on the whole trajectory (X^1_t), $t \in \mathbb{R}$, is that the information about intersections needed to determine $T^k, k \le -1$, can be expressed only by T^0 and $(X^1_t), t \le T^0 \le 0$, since by the definition of T^0, we have that $(X^1_{(-\infty,k-1]})^R \cap (X^1_{[T^0,\infty)})^R = \emptyset$ for all $k \le T^0$. In the sequel, we will slightly abuse the notation. One has to think of the following objects to be defined on an extension of the probability space $(W, \mathcal{W}, P)$, see (2.26) and below. Recall that under the measure $P = \bar{P} \otimes \mathbf{\Lambda}^\varepsilon$, see (2.12), the process $(X^1_t)_{t\in\mathbb{R}}$ is a two-sided d_1-dimensional Brownian motion with $P[X^1_0 = 0] = 1$ which is independent of $(\Lambda_n)_{n\in\mathbb{Z}}$, a two-sided sequence of i.i.d. Bernoulli random variables with success parameter $\varepsilon > 0$, see (2.7). We are interested in large values of p and set

$$L = \left[\frac{p}{3}\right]. \tag{3.60}$$

We introduce a copy $((X^+_t)_{t\in\mathbb{R}}, (\Lambda^+_j)_{j\in\mathbb{Z}})$ of $((X^1_t)_{t\in\mathbb{R}}, (\Lambda_j)_{j\in\mathbb{Z}})$ evolving according to P such that $X^+_t = X^1_{t+p} - X^1_p$ for $t \in [-L, \infty)$, $\Lambda^+_j = \Lambda_{j+p}$ for $j \ge -L$, and such that $((X^+_t)_{t\in(-\infty,-L)}, (\Lambda^+_j)_{j<-L})$ evolves independently of $((X^1_{t+p} -$

$X^1_p)_{t\in(-\infty,-L)}, (\Lambda_{j+p})_{j<-L})$. Moreover, we consider another copy $((X^-_t)_{t\in\mathbb{R}}, (\Lambda^-_j)_{j\in\mathbb{Z}})$ of $((X^1_t)_{t\in\mathbb{R}}, (\Lambda_j)_{j\in\mathbb{Z}})$ which is independent of $((X^+_t)_{t\in\mathbb{R}}, (\Lambda^+_j)_{j\in\mathbb{Z}})$ and evolves according to P such that $X^-_t = X^1_t$ for $t \in (-\infty, L]$, $\Lambda^-_j = \Lambda_j$ for $j \le L-1$, and such that $((X^-_t)_{t\in(L,\infty)}, (\Lambda^-_j)_{j\ge L})$ evolves independently of $((X^1_t)_{t\in(L,\infty)}, (\Lambda_j)_{j\ge L})$. Note that

$$\big((X^1_t)_{t\in\mathbb{R}}, (\Lambda_j)_{j\in\mathbb{Z}}\big) \overset{\text{law}}{=} \big((X^+_t)_{t\in\mathbb{R}}, (\Lambda^+_j)_{j\in\mathbb{Z}}\big) \overset{\text{law}}{=} \big((X^-_t)_{t\in\mathbb{R}}, (\Lambda^-_j)_{j\in\mathbb{Z}}\big) \tag{3.61}$$

and

$$\big((X^+_t)_{t\in\mathbb{R}}, (\Lambda^+_j)_{j\in\mathbb{Z}}\big) \quad \text{is independent of } \big((X^-_t)_{t\in\mathbb{R}}, (\Lambda^-_j)_{j\in\mathbb{Z}}\big). \tag{3.62}$$

The random time T^- is defined like T^0 relatively to $((X^-_t)_{t\in\mathbb{R}}, (\Lambda^-_j)_{j\in\mathbb{Z}})$, and T^+ is the analogue of T^1 attached to $((X^+_t)_{t\in\mathbb{R}}, (\Lambda^+_j)_{j\in\mathbb{Z}})$. The random set $\mathcal{C}^+$ is defined analogously to $\mathcal{C}$ with $((X^+_t)_{t\in\mathbb{R}}, (\Lambda^+_j)_{j\in\mathbb{Z}})$ in place of $((X^1_t)_{t\in\mathbb{R}}, (\Lambda_j)_{j\in\mathbb{Z}})$, see (2.24). We then define

$$\begin{aligned} U &= E^{M^s}[B], \qquad U^- = \psi\big(T^-, (X^-_t)_{t\le 0}, (\Lambda^-_n)_{n\le -1}\big), \\ V &= \big(E^{M^s}[H]\mathbb{1}_{\{0\in\mathcal{C}\}}\big)\circ\theta_p \\ &= \varphi\big(T^1\circ\theta_p, (X^1_{t+p} - X^1_p)_{t\ge 0}, (\Lambda_{n+p})_{n\ge 0}\big)\mathbb{1}_{\{p\in\mathcal{C}^+\}}, \\ V^+ &= \varphi\big(T^+, (X^+_t)_{t\ge 0}, (\Lambda^+_n)_{n\ge 0}\big)\mathbb{1}_{\{0\in\mathcal{C}^+\}}. \end{aligned} \tag{3.63}$$

By construction, see in particular (3.61) and (3.62), we have that $U \overset{\text{law}}{=} U^-$ and, due to the invariance of P under the shift θ_p, also $V \overset{\text{law}}{=} V^+$, but U^- and V^+ are now independent. For $p \ge 1$,

$$\begin{aligned} E^{Q^s}\big[(H\mathbb{1}_{\{0\in\mathcal{C}\}})\circ\Theta_p\cdot B\big] &\overset{(3.58)}{=} E^P[VU] \\ &= E^P[V^+U^-] + E^P\big[V^+(U-U^-)\big] \\ &\quad + E^P\big[(V-V^+)U\big]. \end{aligned}$$

Note that the second line vanishes because of the independence mentioned above and the fact that

$$\begin{aligned} E^P[V^+] &= E^P[V] \overset{(3.58)}{=} E^{Q^s}[H\mathbb{1}_{\{0\in\mathcal{C}\}}] \overset{(2.31)}{=} E^{\hat{Q}^s}[H]E^{\hat{P}}\big[T^1\big]^{-1} \\ &\overset{(2.60)}{=} E^{Q^s}[\mathcal{Z}] \overset{(3.47)}{=} 0. \end{aligned}$$

Therefore, after recalling that $\|U\|_2 \le \|B\|_2 = 1$, we find with Hölder's inequality:

$$E^{Q^s}\big[(H\mathbb{1}_{\{0\in\mathcal{C}\}})\circ\Theta_p\cdot B\big] \le \|V^+\|_4\|U-U^-\|_{4/3} + \|V-V^+\|_2. \tag{3.64}$$

Due to the stationarity of θ_1 under P and Jensen's inequality, we easily obtain that

$$\|V^+\|_4 = \|V\|_4 \le E^{Q^s}\big[|H|^4\mathbb{1}_{\{0\in\mathcal{C}\}}\big]^{1/4}$$

$$= E^{\hat{Q}^s}\big[|H|^4\big]^{1/4} P[0 \in \mathcal{C}]^{1/4} \le E^{\hat{Q}^s}\big[|H|^4\big]^{1/4}.$$

From the definition of H, see (3.54), and Remark (2.7) it then follows that

$$\|V^+\|_4 = \|V\|_4 \le E^{\hat{P}\times K_0}\big[|\chi_{T^1}|^4\big]^{1/4} + vE^{\hat{P}}\big[(T^1)^4\big]^{1/4} \overset{(3.4),(3.6)}{<} \infty. \tag{3.65}$$

In view of the definitions (3.63), we see that

$$\|V - V^+\|_2 \le \big\|\big(|V| + |V^+|\big)\big(\mathbb{1}_{\{T^+\neq T^1\circ\theta_p\}} + |\mathbb{1}_{\{p\in\mathcal{C}\}} - \mathbb{1}_{\{0\in\mathcal{C}^+\}}|\big)\big\|_2. \tag{3.66}$$

Since by the stationarity of θ_1 under P and the identity in law (3.61), $P[\{p \in \mathcal{C}\}\setminus\{0 \in \mathcal{C}^+\}]$ is equal to $P[\{0 \in \mathcal{C}^+\}\setminus\{p \in \mathcal{C}\}]$, an application of Cauchy–Schwarz' inequality to the right-hand side of (3.66) shows that

$$\|V - V^+\|_2 \le 2\|V\|_4\big(P\big[T^+ \neq T^1 \circ \theta_p\big]^{1/4} + 2P\big[\{p \in \mathcal{C}\} \setminus \{0 \in \mathcal{C}^+\}\big]^{1/4}\big). \tag{3.67}$$

Since (X_t^+, Λ_n^+) and $(X_t^1, \Lambda_n) \circ \theta_p$ coincides for $t \in [-L, \infty)$, $n \ge -L$, with (2.24), we see that for large p, the events $\{T^+ \neq T^1 \circ \theta_p\}$ and $\{p \in \mathcal{C}\} \setminus \{0 \in \mathcal{C}^+\}$ are both included in

$$\big\{\big(X^+_{(-\infty,-L]}\big)^R \cap \big(X^+_{[0,\infty)}\big)^R \neq \emptyset\big\} \cup \big\{\big((X^1_\cdot \circ \theta_p)_{(-\infty,-L]}\big)^R \cap \big((X^1_\cdot \circ \theta_p)_{[0,\infty)}\big)^R \neq \emptyset\big\},$$

and so, together with (3.67), we find using the stationarity once again that

$$\|V - V^+\|_2 \le c\|V\|_4 P\big[\big(X^1_{(-\infty,0]}\big)^R \cap \big(X^1_{[L,\infty)}\big)^R \neq \emptyset\big]^{1/4}. \tag{3.68}$$

By analogous arguments as above we also find that

$$\begin{aligned}
&\|U - U^-\|_{4/3} \\
&\quad \le \big\|\big(|U| + |U^-|\big)\mathbb{1}_{\{T^0\neq T^-\}}\big\|_{4/3} \\
&\quad \le \big\|\big(|U| + |U^-|\big)\big(\mathbb{1}\big\{\big(X^-_{(-\infty,0]}\big)^R \cap \big(X^-_{[L,\infty)}\big)^R \neq \emptyset\big\} \\
&\qquad + \mathbb{1}\big\{\big(X^1_{(-\infty,0]}\big)^R \cap \big(X^1_{[L,\infty)}\big)^R \neq \emptyset\big\}\big)\big\|_{4/3} \\
&\quad \le cP\big[\big(X^1_{(-\infty,0]}\big)^R \cap \big(X^1_{[L,\infty)}\big)^R \neq \emptyset\big]^{1/4},
\end{aligned} \tag{3.69}$$

where we used Hölder's inequality and $\|U^-\|_2 = \|U\|_2 \le \|B\|_2 = 1$ in the last inequality. Collecting (3.64), (3.65), (3.68), and (3.69), we finally find

$$\begin{aligned}
\big\|E^{Q^s}\big[(H\mathbb{1}_{\{0\in\mathcal{C}\}}) \circ \Theta_p \mid \mathcal{G}_0\big]\big\|_2 \;&\le\; c\|V\|_4 P\big[\big(X^1_{(-\infty,0]}\big)^R \cap \big(X^1_{[L,\infty)}\big)^R \neq \emptyset\big]^{1/4} \\
&\overset{(2.39)}{\le}\; c\|V\|_4\, p^{-\frac{d_1-4}{8}}.
\end{aligned}$$

This quantity is summable in p, since $d_1 \ge 13$. This finishes the proof of (3.53) and thus of Theorem 3.8. □

Remark 3.10 In the next section we will strengthen Theorems 3.3 and 3.8 to central limit theorems under the quenched measure. In the literature only few results on quenched invariance principles for diffusions in random environment are available. One result is due to Sznitman–Zeitouni [36], who consider small perturbations of Brownian motion, and a second situation in which a quenched central limit theorem holds is discussed in Osada [21]. The latter result is attained with the technique of the *environment viewed from the particle.*

4 Central Limit Theorem under the Quenched Measure

We are going to show how one can improve the results of Sect. 3 to central limit theorems under the quenched measure $P_{0,\omega}$. We use an idea of Bolthausen and Sznitman, see Lemma 4 in [3], to turn the *annealed* invariance principle into a *quenched* invariance principle by bounding certain variances through the control of intersections of two independent paths. For this purpose, we do not require an explicit invariant measure for the process of the environment viewed from the particle or the control of moments of certain regeneration times, see, for instance, [1, 24, 25] in the discrete setting. We recall the definition of v in (2.66).

Theorem 4.1 *Assume $d_1 \geq 7$ and (3.1), or $d_1 \geq 13$. Then for $\mathbb{P}$-a.e. ω, under the measure $P_{0,\omega}$, the $C(\mathbb{R}_+, \mathbb{R}^d)$-valued random variables*

$$B^r_{\cdot} \overset{\text{def}}{=} \frac{1}{\sqrt{r}}(X_{r\cdot} - vr\cdot), \qquad r > 0,$$

converge weakly to a Brownian motion $B_{\cdot}$ with covariance matrix A given in Theorem 3.3 and 3.8 respectively as $r \to \infty$.

Proof By similar arguments as at the beginning of the proof of Theorem 3.3, see (3.16) and (3.18), and the identity in law in (1) of Theorem 2.2, we can see that it suffices to show that

$$\begin{aligned}&\text{for } \mathbb{P}\text{-a.e. } \omega, \text{ under the measure } P \times K_{0,\omega}, \text{ the } C(\mathbb{R}_+, \mathbb{R}^d)\text{-valued random}\\ &\text{variables } \beta^n_{\cdot} \overset{\text{def}}{=} \tfrac{1}{\sqrt{n}}\{\chi_{[n\cdot]} + (n\cdot - [n\cdot])(\chi_{[n\cdot]+1} - \chi_{[n\cdot]}) - vn\cdot\},\ n \geq 1,\\ &\text{converge weakly to } B_{\cdot} \text{ as } n \to \infty.\end{aligned} \tag{4.1}$$

From the proofs of Theorem 3.3 and 3.8, see in particular (3.16), (3.18), (3.45), and (3.46), we know that

$$\beta^n_{\cdot} \to B_{\cdot} \text{ in law under } P \times K_0 \text{ as } n \to \infty. \tag{4.2}$$

From the proof of Lemma 4.1 in [3] we see that (4.1) follows from (4.2) and a variance calculation. Let us introduce, for $T > 0$, the space of continuous $\mathbb{R}^d$-valued functions on $[0, T]$ denoted by $C([0, T], \mathbb{R}^d)$ and equipped with the distance

$$d_T(g, g') \overset{\text{def}}{=} \sup_{t \leq T} \big|g(t) - g'(t)\big| \wedge 1. \tag{4.3}$$

The proof of Lemma 4.1 in [3] shows us that (4.1) follows once we prove that, for all $T>0$, $\xi\in(1,2]$, and all bounded Lipschitz functions F on $C([0,T],\mathbb{R}^d)$,

$$\sum_m \mathrm{Var}_{\mathbb{P}}\big(E^{P\times K_{0,\omega}}\big[F\big(\beta_\cdot^{[\xi^m]}\big)\big]\big)<\infty \tag{4.4}$$

(with a slight abuse of notation). For this purpose, we need some further notation. Given an environment ω, we consider two independent copies $((\chi_t)_{t\ge0},(\Lambda_n)_{n\ge0})$ and $((\tilde\chi_t)_{t\ge0},(\tilde\Lambda_n)_{n\ge0})$ evolving according to $P\times K_{0,\omega}$. The corresponding first d_1 components of $\chi_\cdot$ and $\tilde\chi_\cdot$, denoted by $X_\cdot^1$ and $\tilde X_\cdot^1$, are then two independent d_1-dimensional Brownian motions. We also introduce the corresponding polygonal interpolations $\beta_\cdot^n$ and $\tilde\beta_\cdot^n$ defined as in (4.1). By $\mathcal{D}$ we denote the set of one-sided cut times attached to $((X_t^1)_{t\in\mathbb{R}},(\Lambda_j)_{j\in\mathbb{Z}})$ defined via

$$\mathcal{D}=\big\{k\ge1\ \big|\ \big(X^1_{[0,k-1]}\big)^R\cap\big(X^1_{[k,\infty)}\big)^R=\emptyset \text{ and } \Lambda_{k-1}=1\big\}. \tag{4.5}$$

$\tilde{\mathcal{D}}$ is defined analogously and attached to $((\tilde X_t^1)_{t\in\mathbb{R}},(\tilde\Lambda_j)_{j\in\mathbb{Z}})$. We then pick

$$\xi\in(1,2],\quad 0<\mu<\nu<\frac12,$$

and, for $m\ge1$, we define $n=[\xi^m]$, as well as

$$\rho_m\overset{\text{def}}{=}\inf\big\{\mathcal{D}\cap[n^\mu,\infty)\big\}<\infty,\qquad P\text{-a.s.}\qquad(\text{see (2.28)}),$$

and $\tilde\rho_m$ as the corresponding variable attached to $((\tilde X_t^1)_{t\in\mathbb{R}},(\tilde\Lambda_j)_{j\in\mathbb{Z}})$. In order to take advantage of decoupling effects, we will consider the event

$$\mathcal{A}_m=\big\{\rho_m\vee\tilde\rho_m\le n^\nu,\ \big(X^1_{[0,\infty)}\big)^R\cap\big(\tilde X^1_{[n^\mu,\infty)}\big)^R=\emptyset,\ \big(X^1_{[n^\mu,\infty)}\big)^R\cap\big(\tilde X^1_{[0,\infty)}\big)^R=\emptyset\big\}.$$

We are now ready to prove (4.4). Without loss of generality, we assume the Lipschitz constant and the absolute value of F to be bounded by 1. For the remainder of the proof, we write E and E_ω for the expectations under the measure $P\times K_0$ and $P\times K_{0,\omega}$, respectively. For $m\ge1$, we have

$$\begin{aligned}\mathrm{Var}_{\mathbb{P}}\big(E_\omega\big[F(\beta_\cdot^n)\big]\big)&=\mathbb{E}\big[E_\omega\otimes E_\omega\big[F\big(\beta_\cdot^n\big)F\big(\tilde\beta_\cdot^n\big)\big]\big]-E\otimes E\big[F\big(\beta_\cdot^n\big)F\big(\tilde\beta_\cdot^n\big)\big]\\&=\mathbb{E}\big[E_\omega\otimes E_\omega\big[F\big(\beta_\cdot^n\big)F\big(\tilde\beta_\cdot^n\big),\mathcal{A}_m\big]\big]\\&\quad-E\otimes E\big[F\big(\beta_\cdot^n\big)F\big(\tilde\beta_\cdot^n\big),\mathcal{A}_m\big]+d_m\end{aligned}$$

with

$$|d_m|\le 2P\otimes P\big[\mathcal{A}_m^c\big]. \tag{4.6}$$

Using that F is bounded and Lipschitz and $d_T(\cdot,\cdot)\le1$, we obtain that the difference of the first two terms in the last line above (4.6) is equal to

$$\mathbb{E}\big[E_\omega\otimes E_\omega\big[F\big(\beta^n_{\cdot+\frac{\rho_m}{n}}-\beta^n_{\frac{\rho_m}{n}}\big)F\big(\tilde\beta^n_{\cdot+\frac{\tilde\rho_m}{n}}-\tilde\beta^n_{\frac{\tilde\rho_m}{n}}\big),\mathcal{A}_m\big]\big]$$

$$
\begin{aligned}
&- E\otimes E\big[F\big(\beta^n_{\cdot+\frac{\rho_m}{n}}-\beta^n_{\frac{\rho_m}{n}}\big)F\big(\tilde\beta^n_{\cdot+\frac{\tilde\rho_m}{n}}-\tilde\beta^n_{\frac{\tilde\rho_m}{n}}\big),\mathcal{A}_m\big]+\Delta^3_m\\
&=:\Delta^1_m-\Delta^2_m+\Delta^3_m
\end{aligned}
\tag{4.7}
$$

with

$$
\Delta^3_m\le 6E\otimes E\big[d_T\big(\beta^n_{\cdot+\frac{\rho_m}{n}}-\beta^n_{\frac{\rho_m}{n}},\beta^n_\cdot\big),\mathcal{A}_m\big].
$$

We first want to show that

$$
\Delta^1_m=\Delta^2_m. \tag{4.8}
$$

For each $\omega\in\Omega$ and fixed samples $(w,\lambda_\cdot)$ and $(\tilde w,\tilde\lambda_\cdot)$ of $(X^1_\cdot,\Lambda_\cdot)$ and $(\tilde X^1_\cdot,\tilde\Lambda_\cdot)$, respectively, under $K_{0,\omega}(w,\lambda_\cdot)\otimes K_{0,\omega}(\tilde w_\cdot,\tilde\lambda_\cdot)$, the processes $\beta^n_\cdot$ and $\tilde\beta^n_\cdot$ are independent, and hence with the help of Fubini's Theorem we can write

$$
\Delta^1_m=E^P\otimes E^P\big[\mathbb{E}\big[E^{K_{0,\omega}}\big[F\big(\beta^n_{\cdot+\frac{\rho_m}{n}}-\beta^n_{\frac{\rho_m}{n}}\big)\big]E^{K_{0,\omega}}\big[F\big(\tilde\beta^n_{\cdot+\frac{\tilde\rho_m}{n}}-\tilde\beta^n_{\frac{\tilde\rho_m}{n}}\big)\big]\big],\mathcal{A}_m\big].
$$

By similar arguments to those leading to (2.53) we obtain that, for each $(w,\lambda_\cdot)\in W$,

$$
\begin{aligned}
&E^{K_{0,\omega}}\big[F\big(\beta^n_{\cdot+\frac{\rho_m}{n}}-\beta^n_{\frac{\rho_m}{n}}\big)\big]\\
&\quad=\int_{\mathbb{R}^{d_2}}\frac{dy}{\mathrm{vol}(d_2)}E^{K_{0,\omega}}\big[\mathbb{1}\big\{y\in B^{d_2}_1\big(X^2_{\rho_m-1}\big)\big\}\big]\\
&\qquad\times E^{K_{0,\bar\omega}\circ\theta_{\rho_m}}\big[F\big(\beta^n_\cdot-\beta^n_0\big)\circ\theta_{\rho_m}\big]
\end{aligned}
\tag{4.9}
$$

with $\bar\omega=\tau_{(w(\rho_m),y)}(\omega)$. From (4) of Theorem 2.2 it follows that the first expectation in the second line of (4.9) is measurable with respect to $\mathcal{H}_{(w([0,\rho_m-1]))\times\mathbb{R}^{d_2}}$, see (1.4), whereas the second expectation is $\mathcal{H}_{(w([\rho_m,\infty)))\times\mathbb{R}^{d_2}}$-measurable. With these considerations in mind, we find with Fubini's Theorem and finite range dependence, see (1.5), that on $\mathcal{A}_m$,

$$
\begin{aligned}
&\mathbb{E}\big[E^{K_{0,\omega}}\big[F\big(\beta^n_{\cdot+\frac{\rho_m}{n}}-\beta^n_{\frac{\rho_m}{n}}\big)\big]E^{K_{0,\omega}}\big[F\big(\tilde\beta^n_{\cdot+\frac{\tilde\rho_m}{n}}-\tilde\beta^n_{\frac{\tilde\rho_m}{n}}\big)\big]\big]\\
&\quad=\int_{\mathbb{R}^{d_2}}\int_{\mathbb{R}^{d_2}}\frac{dy_1\,dy_2}{\mathrm{vol}(d_2)^2}\mathbb{E}\big[E^{K_{0,\omega}}\big[\mathbb{1}\big\{y_1\in B^{d_2}_1\big(X^2_{\rho_m-1}\big)\big\}\big]\\
&\qquad\times E^{K_{0,\omega}}\big[\mathbb{1}\big\{y_2\in B^{d_2}_1\big(\tilde X^2_{\tilde\rho_m-1}\big)\big\}\big]\big]\mathbb{E}\big[E^{K_{0,\bar\omega}\circ\theta_{\rho_m}}\big[F\big(\beta^n_\cdot-\beta^n_0\big)\circ\theta_{\rho_m}\big]\big]\\
&\qquad\times\mathbb{E}\big[E^{K_{0,\tilde\omega}\circ\theta_{\tilde\rho_m}}\big[F\big(\tilde\beta^n_\cdot-\tilde\beta^n_0\big)\circ\theta_{\tilde\rho_m}\big]\big]
\end{aligned}
\tag{4.10}
$$

with $\bar\omega=\tau_{(w(\rho_m),y_1)}(\omega)$, $\tilde\omega=\tau_{(w(\tilde\rho_m),y_2)}(\omega)$. Because of the stationarity of the environment, the last two $\mathbb{P}$-expectations above are in fact independent of y_1 respectively y_2 so that an application of Fubini's Theorem shows us that (4.10) equals

$$
E^{K_0\circ\theta_{\rho_m}}\big[F\big(\beta^n_\cdot-\beta^n_0\big)\circ\theta_{\rho_m}\big]E^{K_0\circ\theta_{\tilde\rho_m}}\big[F\big(\tilde\beta^n_\cdot-\tilde\beta^n_0\big)\circ\theta_{\tilde\rho_m}\big].
$$

Analogously we also find that

$$
E^{K_0}\big[F\big(\beta^n_{\cdot+\frac{\rho_m}{n}}-\beta^n_{\frac{\rho_m}{n}}\big)\big]=E^{K_0\circ\theta_{\rho_m}}\big[F\big(\beta^n_\cdot-\beta^n_0\big)\circ\theta_{\rho_m}\big],
$$

and the same holds true if we replace $\beta^n_{\cdot}$, ρ_m by $\tilde{\beta}^n_{\cdot}$, $\tilde{\rho}_m$. This concludes the proof of (4.8). We now come to the control of Δ^3_m. Noting that on $\mathcal{A}_m$, E_ω-a.s.

$$d_T\big(\beta^n_{\cdot+\frac{\rho_m}{n}} - \beta^n_{\frac{\rho_m}{n}}, \beta^n_{\cdot}\big) \le \sup_{\substack{0\le s<t\le Tn+1\\ |t-s|\le n^\nu}} \frac{1}{\sqrt{n}}|\chi_t - \chi_s| + \sup_{t\le n^\nu} \frac{1}{\sqrt{n}}|\chi_t|,$$

we find by using (2.13) and the fact that

$$E_\omega\Big[\sup_{\substack{0\le s<t\le Tn+1\\ |t-s|\le n^\nu}} \big|W_t - W_s\big|\Big]$$
$$\le c(T)n^{1/4+\nu/4} E_\omega\Big[\sup_{0\le s\le t\le 1} \frac{|W_t - W_s|}{|t-s|^{1/4}}\Big] \le c(T)n^{1/4+\nu/4},$$

where the last inequality follows from an application of Fernique's Theorem, see [7] on p. 14, which ensures the existence of the exponential moment of $\eta \sup_{0\le s\le t\le 1} |W_t - W_s|/|t-s|^{1/4}$ for a certain constant $\eta > 0$, that

$$\Delta^3_m \le \frac{c(T)}{\sqrt{n}}\big(n^\nu + n^{1/4+\nu/2}\big) \le c(T)n^{\nu/2-1/4},$$

and hence $\sum_m \Delta^3_m < \infty$, recalling that $n = [\xi^m]$. It remains to show that

$$\sum_m P\otimes P\big[\mathcal{A}^c_m\big] < \infty. \tag{4.11}$$

Indeed, we find that

$$P\otimes P\big[\big(X^1_{[0,\infty)}\big)^R \cap \big(\tilde{X}^1_{[n^\mu,\infty)}\big)^R \neq \emptyset\big] \overset{(2.39)}{\le} cn^{-\mu\frac{d_1-4}{2}}, \tag{4.12}$$

and moreover, since the random set $\mathcal{C}\cap\mathbb{N}$ is contained in $\mathcal{D}$, see (2.24) and (4.5), we have that P-a.s., $\rho_m - n^\mu \le T^1 \circ \theta_{[n^\mu]}$, and hence from the stationarity of θ_1 under P it follows that for large m,

$$P\big[\rho_m > n^\nu\big] \le P\big[T^1 > n^\nu - n^\mu\big] \overset{(2.33)}{\le} c(\varepsilon)\big(\log n^\nu\big)^{1+\frac{d_1-4}{2}} n^{-\nu\frac{d_1-4}{2}} \le e^{-c(\varepsilon)m}. \tag{4.13}$$

Combining (4.12) and (4.13), we deduce (4.11). □

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Appendix: Two Central Limit Theorems for Martingales

For $T > 0$ and integer $d \ge 1$, we denote by $D([0,T],\mathbb{R}^d)$ the space of all $\mathbb{R}^d$-valued càdlàg functions on $[0,T]$ endowed with the Skorokhod metric. The space of all continuous functions on $[0,T]$ with values in $\mathbb{R}^d$ is denoted with $C([0,T],\mathbb{R}^d)$ and is equipped with the supnorm. See Chaps. 2 and 3 of [2] for an extensive discussion of the above mentioned function spaces.

Theorem 5.1 *X_n, $\mathcal{F}_n \overset{\text{def}}{=} \sigma(X_k,\ k \le n)$, $n \ge 0$, is an $\mathbb{R}^d$-valued sequence of square integrable martingale differences on a probability space $(\Omega,\mathcal{F},P)$, i.e., $E[|X_n|^2] < \infty$ and $E[X_n|\mathcal{F}_{n-1}] = 0$. Let Γ be a symmetric, nonnegative definite $d \times d$-matrix and $S_n(\cdot) \overset{\text{def}}{=} \sum_{k=1}^{[n\cdot]} X_k$. Assume that*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{[ns]} E\big[X_k X_k^t \,\big|\, \mathcal{F}_{k-1}\big] = s\Gamma \quad \textit{in probability} \tag{5.1}$$

for each $s \in \mathbb{R}_+$ and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} E\big[\big|X_k\big|^2 \mathbb{1}\big\{|X_k| \ge \varepsilon\sqrt{n}\big\} \,\big|\, \mathcal{F}_{k-1}\big] = 0 \quad \textit{in probability} \tag{5.2}$$

for each $\varepsilon > 0$. Then the $C(\mathbb{R}_+,\mathbb{R}^d)$-valued random variables

$$\frac{1}{\sqrt{n}}\bar{S}_n(\cdot) \overset{\text{def}}{=} \frac{1}{\sqrt{n}}\big\{S_n(\cdot) + \big(n\cdot - [n\cdot]\big)(X_{[n\cdot]+1} - X_{[n\cdot]})\big\}, \quad n \ge 1, \tag{5.3}$$

converge weakly to a d-dimensional Brownian motion $B_.$ with covariance matrix Γ as $n \to \infty$.

Proof An application of the invariance principle for a vector-valued, square-integrable martingale difference array which is proved in [23], see Theorem 3, shows us that

$$\frac{1}{\sqrt{n}} S_n(\cdot) \text{ converges weakly to } B_. \text{ on the Skorokhod space } D([0,1],\mathbb{R}^d) \text{ as } n \to \infty.$$

From (18) in the proof of Theorem 3 in [23] we know that, in view of Remark 3.4, also the polygonal interpolation $n^{-1/2}\bar{S}_{n\cdot}$ converges weakly on $D([0,1],\mathbb{R}^d)$ to the same limit. Since the Skorokhod topology relativized to $C([0,1],\mathbb{R}^d)$ coincides with the uniform topology, there we have in fact the weak convergence on $C([0,1],\mathbb{R}^d)$ for the process $n^{-1/2}\bar{S}_{n\cdot}$. Moreover, the identity

$$\frac{1}{\sqrt{n}}\bar{S}_{n\cdot} = \sqrt{M}\frac{1}{\sqrt{nM}}\bar{S}_{nM\frac{\cdot}{M}}, \quad M \ge 1,$$

shows that the process $n^{-1/2}\bar{S}_{n\cdot}$ indeed converges weakly on $C([0,M],\mathbb{R}^d)$. Then the weak convergence on each $C([0,M],\mathbb{R}^d)$ implies the weak convergence on $C(\mathbb{R}_+,\mathbb{R}^d)$, see [38]. $\square$

Theorem 5.2 *X_n, $\mathcal{F}_n \overset{\text{def}}{=} \sigma(X_m, m \le n)$, $n \in \mathbb{Z}$, is an $\mathbb{R}^d$-valued, ergodic stationary sequence of square integrable martingale differences on a probability space*

$(\Omega, \mathcal{F}, P)$, i.e., $E[|X_n|^2] = E[|X_1|^2] < \infty$ and $E[X_n|\mathcal{F}_{n-1}] = 0$. Let $\Gamma \overset{\text{def}}{=} E[X_1 X_1^t]$ and $S_n(\cdot) \overset{\text{def}}{=} \sum_{k=1}^{[n\cdot]} X_k$. Then, the $C(\mathbb{R}_+, \mathbb{R}^d)$-valued random variables

$$\frac{1}{\sqrt{n}} \bar{S}_n(\cdot) \overset{\text{def}}{=} \frac{1}{\sqrt{n}} \big\{ S_n(\cdot) + \big(n \cdot - [n\cdot]\big)(X_{[n\cdot]+1} - X_{[n\cdot]}) \big\}, \quad n \geq 1,$$

converge weakly to a d-dimensional Brownian motion $B_{\cdot}$ with covariance matrix Γ as $n \to \infty$.

Proof Note that, for an $\mathbb{R}^m$-valued function f with $m \geq 1$ such that $f(X_k) \in L^1(P), k \in \mathbb{Z}$, the conditional expectation $E[f(X_k) \mid \mathcal{F}_{k-1}]$ can be written as $\varphi(X_{k-1}, X_{k-2}, \ldots)$ for a measurable function φ, which does not depend on k. By the ergodic theorem we thus find that

$$P\text{-a.s.}, \quad \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{[ns]} E\big[X_k X_k^t \mid \mathcal{F}_{k-1}\big] = s E\big[X_1 X_1^t\big] = s\Gamma.$$

Let $\delta > 0$ be small and choose $N = N(\delta)$ such that

$$E\big[|X_1|^2, |X_1| \geq \varepsilon\sqrt{N}\big] < \delta.$$

Then,

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} E\big[|X_k|^2 \mathbb{1}_{\{|X_k| \geq \varepsilon\sqrt{n}\}} \mid \mathcal{F}_{k-1}\big] &\leq \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} E\big[|X_k|^2 \mathbb{1}_{\{|X_k| \geq \varepsilon\sqrt{N}\}} \mid \mathcal{F}_{k-1}\big] \\ &= E\big[|X_1|^2, |X_1| \geq \varepsilon\sqrt{N}\big] < \delta, \quad P\text{-a.s.}, \end{aligned}$$

where we used the ergodic theorem in the last equality. An application of Theorem 5.1 concludes the proof. □

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